Endogenous depreciation, the replacement problem, and investment-specific technological change

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Tiivistelmä – Referat – Abstract

This thesis examines the effect of investment-specific technological change on the capital replacement decision and depreciation by extending Mukoyama's (2008) study on endogenous depreciation. When allowing investment-specific technological progress to be described either as a fall in the price of capital or as a growth in the relative productivity of new capital, and capital stock to be determined by the producer's optimization, there arise a method to describe obsolescence as a part of depreciation and capital evolution.

The following three key results are shown when assuming that scrapped capital stock has no value. First, the optimal replacement policy is stationary. Second, the acceleration of investment-specific technological progress accelerates capital replacement, hence also obsolescence. Third, whether investment-specific technological progress is modelled as a fall in the price of capital or as a growth in the relative productivity of new capital, does not impact on the optimal replacement policy. A quantitative exercise shows that the first two results seems to hold even if the scrapped capital stock has some positive value. However, if scrapped capital has some value, then the two approaches to model investment-specific technological progress are no longer equivalent.

The adoption of the capital replacement problem for describing depreciation is a promising approach. Even though there does not exist a closed-from solution for the optimal replacement interval, it can be solved (in the stationary case) as a root of a relative simple transcendental function. The rate of depreciation can be explicitly solved, also in the case of non-stationary replacement policy, but that is computationally more difficult. Physical depreciation (wear and tear) can be disentangled from obsolescence insofar as either one is known. Thus, the results still rely on the estimate of physical depreciation.

Avainsanat – Nyckelord – Keywords
Endogenous depreciation, Vintage capital model, Investment-specific technological change

Contents

Notations and Symbols

$O(t, s)$ the output of a plant whose capital stock is installed at time t and whose age is s

$k(t, s)$ the capital stock installed at time t and whose age is s

$p(t)$ the unit price of capital at time t

$q(t)$ the productivity of a new unit of capital at time t

$(T_i)_{i=0}^{\infty}$ a sequence of points in time when capital stock is replaced

$(I_i)_{i=1}^{\infty}$ a sequence of investments,
where I_i is an amount of investment into the latest vintage of capital at time T_i

$(K_i)_{i=0}^{\infty}$ a sequence of capital stocks,
where K_i is just installed capital stock at time T_i (whose age is 0)

$(R_i)_{i=0}^{\infty}$ a sequence of replacement intervals,
where $R_i \stackrel{\text{def}}{=} \Delta T_{i+1} = T_{i+1} - T_i$ is a replacement interval at time T_i

$I(t)$ a continuous extension of $(I_i)_{i=1}^{\infty}$, that is, $I(T_t) = I_t$ for all $t \in 0, 1, 2, ...$

$R(t)$ a continuous extension of $(R_i)_{i=0}^{\infty}$, that is, $R(T_t) = R_t$ for all $t \in 0, 1, 2, ...$

$k(t, s)$[1] a continuous extension of $(e^{-\delta s} K_i)_{i=0}^{\infty}$,
that is, $k(T_t, s) = e^{-\delta s} K_t$ for all $t \in 0, 1, 2, ...$ and for all $s \geq 0$

r the interest rate

δ the physical depreciation (wear and tear) rate

θ the fraction of scrapped capital stock has a value of new capital stock

γ the rate of change of the unit price of capital

λ the rate of change of the productivity of a new unit of capital

A Total Factor Productivity (TFP)

α the output elasticity of capital

d the stationary depreciation[2] rate

$d(t)$ the non-stationary depreciation rate at time t

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[1]This further specification is employed in Sections 5 - 6, albeit the interpretation *"$k(t, s)$ is the capital stock installed at time t and whose age is s"* still holds.

[2]d is also know as the rate of *economic* depreciation. Correspondingly, $d - \delta$ denotes the rate of *obsolescence*, also known as a depreciation due to *replacement*.

1 Introduction

The rate of depreciation plays an important role in measuring capital stock. Incorrect depreciation rates lead to considerable mis-measurement of the aggregate capital stock. There has been discussion that mis-measured capital stocks can partly solve so called production slowdown puzzle (Musso, 2004, p.27, [1]; Mukoyama, 2008, p.521, [2]). By production slowdown puzzle it is meant the sharp decline in the growth rate of productivity in industrialized countries after the late 1960s. Furthermore, the assumption of a constant depreciation rate is challenged (Epstein & Denny, 1980, [3]). The story behind goes as follows. Capital depreciation can be regarded as consisting of two factors: physical depreciation (wear and tear) and obsolescence. Seemingly, obsolescence plays greater role of these two (Sakellaris & Wilson, 2004, p.3-4, [4]). Again, obsolescence is related to technological change, particularly to investment-specific technological change. Investment-specific technological compromises technological change that is embodied in capital. Greenwood et al. (1997) show that the ratio of investment-specific technological change to (Hicks) neutral technological change has increased in the United States from post-war period to 1990s (Greenwood et al., 1997, [5]). So, one can expect that the rate of obsolescence has risen as well. Moreover, there has been a shift to the types of capital that suffer more from obsolescence: "*In recent times, there has been a shift to investment in ICT assets, which are most susceptible to higher rates of obsolescence*" (Sumit Dey & Chowdhuty, ONS, 2008, [6]). In any case, there is an apparent need for more accurate estimates for capital depreciation and more systematic ways to handle obsolescence as a part of depreciation.

The notion of technology is of an abstract nature and thus suffers from *operationalization* difficulties. For this reason, there is no available uncontroversial techniques to measure it. Along with this concern, obsolescence should be linked to technology, either neutral or investment-specific, somehow. Therefore, it may be rather difficult task to conduct an estimate for capital obsolescence by means of standard econometric techniques. That is why, for instance, Greenwood and Krusell favor modeling approach (quantitative theory) over traditional growth accounting when trying to account investment-specific technological progress (Greenwood & Krusell, 2006, [7]). It is foreseen that explicit methods to model obsolescence are needed. This rationalizes the title of the thesis. To capture obsolescence, it should be endogenously modeled, not be taken as an exogenous parameter.

It is given a brief overview on different methodologies of determining capital depreciation. Particularly, the second section discusses the prevailing technique of determining capital depreciation in national accounting, some econometric estimation techniques are presented and two different approaches to model depreciation are introduced. The third section covers more in detail the model constructed by Mukoyama (2008). In the fourth section it is constructed a generalized version of Mukoyama's model. The fifth section encompasses analytical and numerical results of the model presented in the previous section. The sixth section discusses how the results fit into a broader context of macroeconomic modeling. The last section concludes.

2 Capital depreciation: Different approaches

2.1 Types of depreciation

The concept of depreciation has different meanings depending on the context in which it is used. From the viewpoint of National Accounts, the depreciation of fixed assets is defined as the decline of the aggregate capital stock due to the use of this capital in production. Instead of using the concept of depreciation, in National Accounting depreciation is known as the consumption of fixed capital. National Accounts defines the consumption of fixed capital as: "*the decline, during the course of the accounting period, in the current value of the stock of fixed assets owned and used by a producer as a result of physical deterioration, normal obsolescence or normal accidental damage*" (United Nations System of National Accounts, 2008, p.123, [8]). Even more detailed classification can be used for the sources of change in value of capital stock. Ahmad et al. see that the value of capital asset can change due to five reasons: wear and tear, foreseen obsolescence, exhaustion, and other changes that have an effect on demand and supply for the asset and changes in the overall price level (Ahmad et al., OECD, 2005, [9], p.2). Here, the concept of foreseen obsolescence slightly overlap with other reasons (e.g. with exhaustion) as mentioned by Ahmad et al. In fact, for obsolescence there is given a quite broad definition: *"the process whereby a capital good goes out of use, out of date or experiences a decline in tis capacity to generate returns for reasons other than wear and tear and catastrophes"* (Ahmad et al., OECD, 2005, [9], p.9). Therefore, it should be noted that exhaustion, that is, the rate of capital asset retires, is typically contained in obsolescence. Specifically, the type of exhaustion that is due to technological progress is included in obsolescence. For instance, firms in the ICT sector may have to frequently replace their capital due to rapid technological progress in the computer architecture, whose benefits they want to realize.

From the theoretic point of view, the two sources of the value decline in fixed capital are of particular interest: physical depreciation and obsolescence. The physical depreciation of capital refers to wear and tear that deteriorates the production efficiency of the capital. On contrary, the obsolescence of capital is mainly caused by technical progress leading to the fall in the relative productivity of old capital with respect to new capital – Sakellaris and Wilson even view that *"embodied technological change"* is *"synonymous with obsolescence"* (Sakellaris & Wilson, 2004, p.4, [4]). This relative productivity decline leads to earlier capital replacement. Indeed, the relative productivity decline is also reflected in the price of capital as conclude by Hill in a detailed discussion concerning obsolescence (Hill, 1999, [10]). Nevertheless, the distinction of these two forms of depreciation is typically omitted in macroeconomic models but some studies consider explicitly both of these. It is also possible to distinguish different forms of capital and view that these forms of capital are not equally vulnerable to obsolescence and physical depreciation. For instance, Greenwood in his model treats structures and equipment as different forms of capital and regards that the technological change affects equipment only (Greenwood et al., 1997, [5]). In that model depreciation is linked to the technological change, which implies that equipment is suffering from greater depreciation. Further distinctions among the forms of capital can be made. Prucha and Nadiri distinguish R&D capital from physical capital and estimate different depreciation rates for R&D capital and physical capital (Nadiri & Prucha, 1996, [11]).

2.2 Methods

This subsection gives an overview how the rate of capital depreciation can be determined. First, it is described a process how capital depreciation is determined in National Accounts. The explanation is rather undetailed, since there are considerable differences in practices among Institutions of National Accounts. Second, the aim is to shed light on econometric estimation of capital depreciation by introducing two research papers that covers the topic. The subsection is by no means an exhaustive description of different estimation methodologies.

Consumption of fixed capital can be measured directly or indirectly. The direct method is based on exact market valuation of the stock of fixed assets, whereas the indirect method uses approximations. The latter method is the perpetual inventory method (PIM). The perpetual inventory method is recommended method by System of National Accounts and it is predominantly used to estimate the consumption of fixed capital (SNA, 2008, p.124, [8]). The method enables capital stock to be calculated from associated investments flows. The procedure can be summarized as follows. Data for past investments is needed and the life pattern of an investment is specified in two stages. First, a retirement distribution is determined. The distribution describes how the invested capital life-time is spread over its life length mean. This average life length is estimated from data. Second, a type of the depreciation function is chosen. Typically, this is chosen between two convenient alternatives: an arithmetic (straight-line) depreciation function or a geometric depreciation function. The depreciation function determines the path how the capital depreciates over its life-time. Finally, the accumulated net capital stock is calculated by the finite sum of depreciation corrected investments and initial capital stock. For instance, when geometric depreciation function is assumed and the retirement distribution is not taken into account, then the net capital stock in period t, K_t, can be calculated by PIM method as,

$$K_t = (1 - d_0)^t K_0 + \sum_{i=0}^{t} (1 - d_{t-i})^i I_{t-i}, \tag{1}$$

where $(d_i)_{i=0}^t$ and $(I_i)_{i=0}^t$ denote depreciation rates and investments, respectively, over time, and K_0 is an initial capital stock (cf. Sumit Dey & Chowdhuty, ONS, 2008, [6]).

There are differences in practices among Institutions of National Accounts. For instance, the evaluation of capital consumption in the United States Bureau of Economic Analysis is based on the perpetual inventory method for some assets, while depreciation patterns are taken from empirical studies (OECD, 2001, p.100, [12]) for other assets. However, for most assets depreciation patterns are quite arbitrarily determined. That is why many authors have proposed that depreciation rates should be based on econometric studies (e.g. Prucha, 1995; Nadiri & Prucha, 1996, [11]; Hernndez & Maulen, 2005, [13]).

Nadiri and Prucha (1996) estimate depreciation rate of both physical and R&D capital within the framework of factor demand model. The estimation is performed for U.S. total manufacturing sector data. The stocks of physical and R&D capital are assumed to accumulate according to traditional law of motion for capital with a constant depreciation rate. Their general estimation strategy is recursively solve physical and R&D

capital stock with respect to flow of investments and initial capital stock. Obtained time-series for two forms of capital is then plugged into demand equations for other inputs, namely materials and labor. These demand equations are derived consistent way from their framework with additional assumptions on production technology. They adopt "quite general" functional form for cost function, which represents production technology. Finally, they construct time-series for expected output and input prices using a second-order vector autoregressive model. They have data for all variables appearing in aforementioned two demand equations. Therefore, the depreciation rates for both forms of capital can be estimated jointly with other model parameters. Nadiri and Prucha use a numerical algorithm that maximizes a statistical objective function which is eventually based on the Gaussian full information maximum likelihood function. The complexity of estimation procedure stems from the fact that substituted capital stocks in terms of lagged investments depends on different altering number of investments (i.e. number investment lags required to represent capital stock of time t depends on that time parameter t).

Hernndez and Maulen (2005) suggest a method to estimate the rate of depreciation based on a production. They implement the method by fitting a Cobb-Douglas production function to Spanish economy data and by carrying out full maximum likelihood estimation. The method is similar kind of the method proposed by Nadiri and Prucha, but instead of factor demand equations it is used the production function with only two inputs (labor and capital), and depreciation rate is allowed to vary with respect to some predetermined economic variable. For instance, they estimate one regression by assuming that depreciation rate is in a linear relationship with the growth rate of GDP. Since the proposed econometric approach involves highly non-linear relationships, there are some methodological restrictions for the estimation procedure. The main focus of the paper is to provide practical and easier method to carry out the estimation. The new method is basically the second-order approximation of the original problem and the non-linearity is just left to the parameters whereas the variables become linear. Hernndez and Maulen argue that the estimation via production function than via factor demand equation would lead to more robust results. The reason is that the production function is more technically based whereas factor demand equations are more behavioral in nature. In that case the latter approach is more subject to different kinds of specification errors.

2.3 Models

Instead of treating capital depreciation as an exogenous parameter or independent from underlying context, it can be determined endogenously within an appropriate framework. At least two strands in the literature can be distinguished. These vary with respect to the object they relate capital depreciation. The first focus on factors that are under control of a producer such as capital utilization rate and capital maintenance. The second tries to explain depreciation rate by the technological progress, which is an external factor from the viewpoint of a producer.

The first approach relates capital depreciation to capital utilization or capital maintenance. In the former, it is assumed that capital may not be fully utilized. A producer is able to choose capital utilization rate. Typically, it is assumed that depreciation rate is a positive convex function of utilization rate (e.g. Calvo, 1975, [14]; Chatterjee, 2005,

[15]). That is, the more intensively capital is used, the greater is capital depreciation. Moreover, the marginal effect is increasing. It can be also found applications of this methodology in the context of the general equilibrium literature (e.g. Greenwood et al., 1988, [16]). In the case of capital maintenance, it is assumed that in addition to investing in new capital, the producer can also invest in capital maintenance. The more is invested in maintenance, the less is capital depreciation. The idea is that the producer can impede capital depreciation by maintaining capital stock more intensively. Technically, it is just usually assumed that the rate of depreciation is a negative function of maintenance expenditures. Indeed, these two methodologies can be combined as several authors have done (e.g. Fujisaki & Mino, 2009, [17] and Deli, 2016, [18]). That is, capital depreciation depends on capital utilization and capital maintenance, positively and negatively, respectively.

The second approach relates capital depreciation to technological progress. Particularly, the rate of depreciation is affected by investment-specific technological change. Three studies taking this approach are introduced. The last two papers are main references of the thesis.

Musso (2004) constructs a vintage capital two-sector model of unbalanced economic growth (Musso, 2004, [1]). The first sector produces capital structures and consumption goods. The second sector produces capital equipment. Both sectors benefit from neutral technological progress. However, there is embodied (i.e. I-S) technological change only in the second sector. The embodied technological change is modeled by a quality efficiency function that describes the marginal product of capital as a multivariate function of time, vintage and capital type (either equipment or structure). In particularly, the efficiency of capital varies over time. Musso also assumes that in order to retain a certain fraction of old capital it is required to pay maintenance costs. The main result of the paper, based on the simulations of the model, is that the average service-life of equipment has shortened since mid-1960s in the United States.

Greenwood et al. (1997) embed a simple vintage capital model into general equilibrium framework in order to study the role of investment-specific technological change as an engine of economic growth (Greenwood et al., 1997, [5]). Their framework implies a formula[3] to compute the rate of obsolescence for given the rate of physical depreciation and investment-specific technological change (Greenwood et al., 1997, [5], p.361),

$$d = 1 - \frac{q_{t-1}}{q_t}(1-\delta), \tag{2}$$

where d is depreciation rate, δ is physical depreciation rate and q_t reflects the level of investment-specific technology at period t. The formula is obtained as a by-product of an transformation from one presentation of an economy with I-S technological change into another presentation of the same economy (see Appendix B, Greenwood et al., 1997, [5]). Thus, the formula does not steam from the explicit analysis on how IS-technology affects on decisions of economic agents. Their empirical analysis rests on the assumption that Gordon's (1990) price index[4] reflects the productivity in the production of new capital goods, that is, Gordon's index is a good measure of investment-specific technological change, which in the above formula means that Gordon's index equals to $1/q_t$. In the vein

[3]Precisely, the formula reads as $q/q_{-1} = (1-\delta_e)/(1-\tilde{\delta_e})$, where in our terms $\delta_e = \delta$ and $\tilde{\delta_e} = d$.

[4]See Robert Gordon's equipment price index (Gordon, 1990, [19]).

of general equilibrium theory, the model starts by giving comprehensive description of the behavior of agents in the economy. In the production side it is assumed that there are two forms of capital: structures and equipment. Investment-specific technological change is assumed to affect equipment only. Greenwood et al. calibrate the model to U.S. National Income and Product Account data and match up the time-series of investments, labor and consumption with their theoretical counterparts in the model. Solving the model gives the path for neutral productivity.

Mukoyama (2008) constructs a vintage capital model in which depreciation due to obsolescence is endogenously determined and physical depreciation (wear and tear) is exogenously given (Mukoyama, 2008, [2]). The model describes how a producer determines the optimal interval for the replacement of capital stock. The producer tries to maximize the current value of a plant by choosing this replacement interval. Without replacement, capital is assumed to decay physically at a constant rate, and when replaced, it is assumed that scrapped capital stock has a constant value of new capital stock. The unit price of capital falls over time due to investment-specific technological change. The main result is that the optimal length of the replacement interval becomes shorter when the investment-specific technological change accelerates. This earlier capital replacement leads to higher rate of obsolescence.

In this thesis, it is focused on the models that relates depreciation to technological change. Particular emphasis is placed on Mukoyama's (2008) model. The reasoning goes as follows. The models that relate the rate of depreciation to capital utilization or maintenance explain the behavior of physical depreciation. Neither how vigorously producer maintains capital stock nor how intensively producer utilizes existing capital, are directly related to the opportunity-cost in the investment decision produced by technological progress. Since obsolescence is directly linked to technological progress, it can be concluded that these kind of models are not appropriate to explain obsolescence. Certainly, if capital maintenance and/or utilization are linked to the inter-temporal investment decision in which technological progress is taken into account, these factors have also effect on obsolescence. Further, obsolescence is more problematic than physical depreciation in the sense that based on a few studies obsolescence plays greater role in economic depreciation than physical depreciation does (this clearly depends on a type of capital) (Sakellaris & Wilson, 2004, p.3-4, [4]; Greenwood et al., 1997, p.361, [5]). That is why it could be more interesting to focus on the models that relate depreciation to technological change.

3 The replacement problem: Mukoyama (2008)

This section introduces a vintage capital model constructed by Mukoyama (Mukoyama, 2008, [2]). A similar kind vintage capital model also appears in Jovanovic and Rob (1999) (Jovanovic & Rob, 1999, [20]). The model is presented without giving derivation. Full derivation of the model can be found in Appendix A [5]. The assumptions of the model are presented and analyzed. Finally, there is a brief overview how to solve the model.

[5]To appreciate full derivation of the model may be helpful to understand the model presented in the next section.

3.1 The model

Mukoyama (2008) presents a following vintage capital model. A producer maximizes the value of a plant. There is only one factor of production, capital. The output of a plant whose capital stock is installed at time t and whose age is s, is denoted by $O(t,s)$. The capital stock, $k(t,s)$, is determined by the installation time t and its age s. That is why the model is a vintage capital model, namely the efficiency of capital is dependent on the installation time (a vintage of the capital). Moreover, when capital ages it becomes more inefficient due to physical depreciation (wear and tear). An important feature of the model is that there is no capital accumulation. The whole capital stock is regularly replaced to new one. For that, the producer has to choose a replacement interval T. The replacement interval determines how often whole capital stock is replaced to the frontier quality capital. Furthermore, it is assumed that old replaced (scrapped) capital has only a value $\theta \in [0,1]$ of new capital. The value of the plant at time t is denoted by $V(t)$. The problem is characterized by the following optimization,

$$
\begin{aligned}
V(t) = \max_T \Bigg(&\int_0^T e^{-rs} O(t,s)ds \\
&+ e^{-rT}\Big(V(t+T) - p(t+T)k(t+T,0) + \theta p(t+T)k(t,T)\Big)\Bigg).
\end{aligned} \tag{3}
$$

where r is the interest rate and $p(t)$ is the unit price of capital at time t (Mukoyama, 2008, [2]).

For subsequent purposes, it is explicitly listed the assumptions postulated in Mukoyama's model. Two assumptions are imposed on the functional form of the capital. The first states that the capital depreciates physically at a proportional rate δ. The second describes how capital is linked to its vintage (i.e. to an installation time t). Mukoyama sees that this assumption is *"necessary for the replacement decision to be stationary"* (Mukoyama, 2008, [2], p.516). The third assumption deals with the production technology and it restricts the form of production function to be Cobb-Douglas with one input. The total-factor productivity A is a constant and the capital share parameter α lies in $]0,1[$. The fourth is the key assumption. It sates that the unit price of capital falls at a steady rate γ over time, so here $\gamma > 0$. The idea is that this assumption reflects investment-specific technological progress. The assumptions are (Mukoyama, 2008, [2]),

(i). $k(t,s) = e^{-\delta s} k(t,0)$

(ii). $k(t,0) = e^{\frac{1}{1-\alpha}\gamma t}$

(iii). $O(t,s) = Ak(t,s)^{\alpha}$

(iv). $p(t) = e^{-\gamma t}$.

Mukoyama solves the model rather detailed manner. Therefore, it is confined to giving the idea of the solution strategy. The solution of the model is given in more general setting in Appendix B.

First, plug the assumptions (i) - (iv) into the equation 3. Then, the trick is to divide the obtained equation by the term $e^{\frac{\alpha}{1-\alpha}\gamma t}$ and redefine the value function $V(t)$ as

$v = v(t) \stackrel{\text{def}}{=} V(t)e^{-\frac{\alpha}{1-\alpha}\gamma t}$. As the result an equation which does not depend on time t is obtained. This time independence is a special property of Mukoyama's model and the property steams from the exponential functional form assumption. Particularly, the assumption (ii) is imposed precisely for this purpose, that is, to ensure the time independent solution. After these steps, the problem becomes to,

$$v = \max_T \left(\int_0^T Ae^{-(r+\alpha\delta)s} ds + e^{-\left(r-\frac{\alpha}{1-\alpha}\gamma\right)T}(v-1) + \theta e^{-(r+\gamma+\delta)T} \right). \quad (4)$$

Two equations can be derived from this equation. The necessary condition for the existence of a local maximum is that the derivative with respect to T should vanish. On the other hand, if there exists a maximum, then the equation 4 itself should hold in the form in which the maximum operator is absent. It is ended up with two equations and two unknowns, namely T and v. Hence, a solution can be easily derived from this algebraic system of equations. The solution takes a form,

$$\begin{aligned} &Ae^{-(r+\alpha\delta)T} - \frac{e^{-\left(r-\frac{\alpha}{1-\alpha}\gamma\right)T}}{1-e^{-\left(r-\frac{\alpha}{1-\alpha}\gamma\right)T}}\left(r - \frac{\alpha}{1-\alpha}\gamma\right) \\ &\left(A\Big(\frac{1-e^{-(r+\alpha\delta)T}}{r+\alpha\delta}\Big) - 1 + \theta e^{-(r+\gamma+\delta)T}\right) \\ &-\theta(r+\gamma+\delta)e^{-(r+\gamma+\delta)T} = 0. \end{aligned} \quad (5)$$

This equation characterizes the optimal replacement interval T. The optimal T can be implicitly solved from the equation 5 for given parameter values $\gamma, \alpha, \delta, \theta$ *and* r.

Once the optimal replacement interval T is obtained, the depreciation rate can be calculated in the steady-state. If a number of plants is assumed to be a constant over time, then the plant age is uniformly distributed over the interval $[0, T]$. The rate of depreciation can be calculated as (see Appendix C),

$$d = \frac{(1-\theta)\frac{k(t-T,T)}{T}}{\int_0^T \frac{k(t-s,s)}{T} ds} + \delta = \frac{(1-\theta)\phi}{e^{\phi T}-1} + \delta, \quad where\ \phi \stackrel{\text{def}}{=} \delta + \frac{\gamma}{1-\alpha}. \quad (6)$$

4 The replacement problem: Generalized model

In this section a generalized[6] version of Mukoyama's model is derived and solved. Essentially, the model is "generalized" in two ways. First, the economy is not assumed to be in the steady-state in the sense that the optimal replacement interval is a constant over time. Second, instead of assuming some specific functional form for capital stock, it is determined endogenously within the model. The mechanical assumption (ii) concerning the evolution of capital stock is replaced by more behavioral dynamics of capital

[6]The reason why the model can be called a "generalized" version of Mukoyama's model is justified in the derivation of Mukoyama's model (see Appendix A). Thus Mukoyama's model can be seen as a special case of this model.

stock. Capital stock is determined by associated investment flows, physical depreciation and investment-specific technological change. This requires that a producer is able to decide the level of investments in addition to replacement times. Moreover, the way how investment-specific technological change is modeled, is unified with other macroeconomic literature. Investment-specific technological change is incorporated into the model in two ways: through the capital accumulation equation in line with Greenwood and Deli (Greenwood et al., 1997, [5] and Deli, 2016, p.323, [18]), and through the price of capital in line with Mukoyama (2008).

4.1 The model

Consider a producer whose aim is to maximize the present value of a plant. The producer decides points in time when capital stock is replaced and chooses an amount of investment into the latest vintage of capital at the moment of the replacement. Let us denote by $(T_i)_{i=0}^{\infty} = (T_0, T_1, T_2, ...)$ a sequence of points in time when capital stock is replaced and by I_t an amount of investment into the latest vintage of capital at time T_t. Denote by $k(t,s)$[7] capital stock installed at time t and whose age is s. The evolution of capital stock is governed by the following initial value problem,

$$\begin{cases} k(T_t, 0) = q(T_t) I_t \\ \frac{\partial}{\partial s} k(t,s) = -\delta k(t,s) \end{cases} , \tag{7}$$

where a function[8] q describes investment-specific technological change. The first equation explains how just installed capital stock is determined by the level of investment-specific technology and the amount of investment. This[9] kind of approach to model investment-specific technological change is commonly found in the macroeconomic literature (e.g. Fisher, 2006,[21]; Deli, 2016, [18]; Greenwood et al., 1988 & 1997, [16] & [5]). Greenwood et al. model the investment-specific technology in similar fashion, particularly the accumulation equation for equipment type of capital is described similarly. They point out that:*"Changes in q formalize the notion of investment-specific technological change."*(Greenwood et al., 1997,p.345,[5]). The second equation says that capital stock physically depreciates at rate δ with respect to its age s, so being Mukoyama's assumption (i) in a differential equation form. It should be noted that capital stock does not accumulate in the ordinary sense. When capital stock is depicted as a function of (continuous) time, it can be perceived "the piece-wise nature" of the dynamics of capital stock. This is illustrated in Figure 1.

[7] It is assumed that $k(s,t)$ is continuously differentiable on variable s.

[8] In stochastic setting, it can be assumed that q follows some stochastic process. For instance, Greenwood et al. assume that q follows first-order Markov process (Greenwood et al., 1997, p.345,[5]).

[9] The difference is that normally the capital stock accumulates. Hence, the evolution of capital stock typically reads as $\dot{K} = -\delta K + qI$.

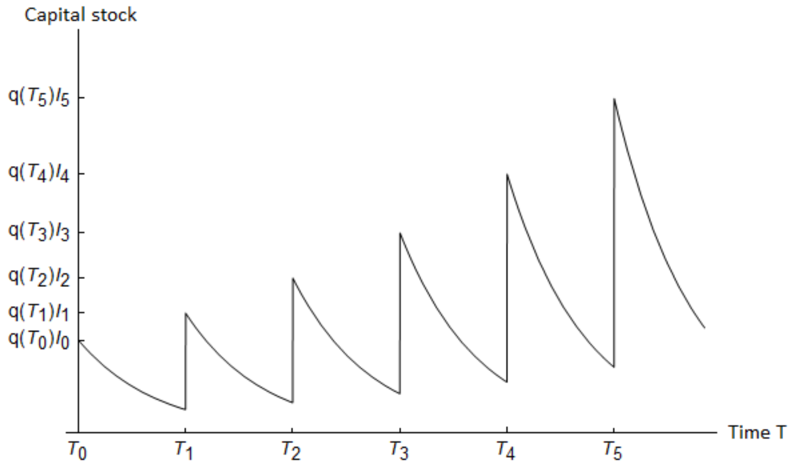

Figure 1. An evolution of capital stock described as a function of time.

After the producer has chosen a sequence of replacement times, say $(T_i)_{i=0}^{\infty}$, the discounted output from the period $]T_t, T_{t+1}]$, whose capital has been just installed, is obtained by summing continuously the discounted outputs,

$$\int_0^{\Delta T_{t+1}} e^{-rs} O(T_t, s) ds, \tag{8}$$

where $r > 0$ denotes the interest rate and $O(t, s)$ denotes the output of a plant whose capital stock is installed at time t and whose age is s. Note that it is integrated over the age of capital stock. The producer invest an amount I_{t+1} into the frontier-quality capital at the end of the period $]T_t, T_{t+1}]$. The total cost of investment also depends on the current price of capital,

$$p(T_{t+1}) I_{t+1}. \tag{9}$$

On the other hand, capital stock is no longer such a valuable as it was in the beginning of the period. In order to get the *value* of capital stock, capital stock should be divided by the factor $q(T_t)$ (see Appendix D). If it is assumed that the fraction θ of the value of capital stock is retrieved back to the producer, then the total rebate of scrapped capital stock is,

$$\theta p(T_{t+1}) \frac{k(T_t, \Delta T_{t+1})}{q(T_t)}. \tag{10}$$

As in Mukoyama (2008), the quantity $(1-\theta)$ measures here "an irreversibility of capital". Parameter θ should not include the aspect of physical depreciation of capital stock, since this is already taken into account in the term (10).

The terms (8) - (10) are exploited in the determination of inter-temporal flow of profits. The profit from the period $]T_t, T_{t+1}]$ is essentially same as the sum of the output and the rebate of scrapped capital stock (i.e. the terms (8) and (10)) minus the cost of investment(i.e. the term (9)). However, the flow of profits from different periods must be discounted adequately. For this purpose, let us describe the (present) value of plant at time t by a function $V(t)$. The value of plant is determined by the discounted flow of profits. Let us consider the value of plant at the beginning of the period, that is at T_t. In that case, the value of plant is determined by,

$$V(T_t) = \max_{(T_i)_{i=t+1}^{\infty},(I_i)_{i=t+1}^{\infty}} \sum_{i=t}^{\infty} \Bigg(e^{-r(T_i-T_t)} \int_0^{\Delta T_{i+1}} e^{-rs} O(T_i,s)ds \tag{11}$$
$$+e^{-r(T_{i+1}-T_t)} \Big(-p(T_{i+1})I_{i+1} + \theta p(T_{i+1}) \frac{k(T_i,\Delta T_{i+1})}{q(T_i)} \Big) \Bigg)$$
$$s.t. \quad for\ all\ i\ it\ holds\ that\ T_{i+1} > T_i\ and\ I_i \geq 0.$$

This equation together with the evolution of capital stock (7) formulate the replacement problem. The initial value T_0 is given, but I_0 is chosen such that $V(T_0)$ is maximized. The budget constraint is excluded because the investment policy is not of primary interest, and the inclusion would complicate the analysis. However, in the presence of I-S technological change, the exclusion of the budget constrain may induce too rapid growth of investments and thus leading to further difficulties in the convergence of the series. It turns out that this not a problem if interest rate r is set to "sufficiently" high level (see Proposition 1).

The solution for the replacement problem would offer an interesting insight into how investment-specific technological change affects to the life cycle of capital stock. Again, this kind of information can be used to calculate the rate of depreciation.

4.2 Solution

The initial value problem (7) has a simple solution (see Appendix E), hence the problem can be reduced to finding a pair of sequences that maximize the functional (11). This type of optimization problem is encountered in a wide range of economic problems (infinite horizon consumer problems, the general equilibrium modeling, etc.) and mathematically the problem can be seen as a discrete calculus of variations problem. Some solution methods (e.g. Cadzow, 1970, [22]) leads to solving non-linear difference equations, which may imply further difficulties or need for approximations. Other solution methods include the associated Lagrangian by means of which the problem can be solved.

Here, it is taken a bit different approach. The problem is first converted into a discrete optimal control problem and then solved in the vein of dynamic programming. This is carried out in three steps: the problem is converted, a corresponding Bellman equation is derived and the optimal paths for control variables are solved by using Bellman equation. At the result of that, it is obtained Euler equations for replacement and investment policies. Those are exploited in both analytical derivations and numerical computations.

4.2.1 Step 1: Convert the problem into a discrete optimal control problem

The problem can be converted into an infinite horizon discrete optimal control problem. First, the evolution of capital stock, which is described as an initial value problem, should be solved and the solution should be rephrased in terms of discrete state variable. In Appendix E it is shown that $k(T_t,s) = e^{-\delta s} q(T_t) I_t$ is the only solution of the initial value problem (7). Define a new state variable K_t, capital stock at time T_t, by $K_t \stackrel{\text{def}}{=} k(T_t, 0)$, and rephrase (by forwarding one period) the "discrete" evolution of the capital stock as,

$$K_{t+1} = q(T_{t+1}) I_{t+1}. \tag{12}$$

Further, the term $k(T_t, \Delta T_{t+1})$ in the objective function must be replaced by $e^{-\delta \Delta T_{t+1}} K_t$. The evolution of replacement times must be reformulated. Let us invoke a new control variable R_t, which describes the replacement interval at time T_t (one can think it as $R_t = \Delta T_{t+1}$). In that case the variable T_t takes a role of state variable and a state equation can be formulated as,

$$T_{t+1} = T_t + R_t. \tag{13}$$

Consequently, there are two state variables K_t and T_t with two control variables I_t and R_t. Given the state equations (12) and (13) for state variables, the problem can be rephrased by a discrete optimal control problem,

$$\begin{aligned} V(T_t, K_t) = \max_{(R_i)_{i=t}^{\infty}, (I_i)_{i=t+1}^{\infty}} \sum_{i=t}^{\infty} \Bigg(& e^{-r(T_i - T_t)} \int_0^{R_i} e^{-rs} O(T_i, s) ds \\ & + e^{-r(T_{i+1} - T_t)} \Big(- p(T_{i+1}) I_{i+1} + \theta p(T_{i+1}) \frac{e^{-\delta R_i} K_i}{q(T_i)} \Big) \Bigg) \\ & s.t. \quad for\ all\ i\ it\ holds\ that\ R_i > 0\ and\ I_i \geq 0. \end{aligned} \tag{14}$$

There are now two initial values, T_0 and K_0, to be determined, whereas there is only one initial value, T_0, in the original problem. Hence, one must bear in mind that the initial capital stock K_0 must be set to the optimal level (from the viewpoint of the producer) rather than regard it as a free value.

4.2.2 Step 2: Derive Bellman equation

In the vein of *Bellman's Principle of Optimality*, the dynamic optimization problem is broken into simpler subproblems. For this goal, Eq. (14) should be modified such that the value of plant is represented in terms of current profit and the future value of plant. A starting point is to extract the first term from the series,

$$\begin{aligned} & V\big(T_t, K_t\big) = \max_{(R_i)_{i=t}^{\infty}, (I_i)_{i=t+1}^{\infty}} e^{-r(T_t - T_t)} \int_0^{R_t} e^{-rs} O(T_t, s) ds \\ & + e^{-r(T_{t+1} - T_t)} \Big(- p(T_{t+1}) I_{t+1} + \theta p(T_{t+1}) \frac{e^{-\delta R_t} K_t}{q(T_t)} \Big) \\ & + \sum_{i=t+1}^{\infty} \Bigg(e^{-r(T_i - T_t)} \int_0^{R_i} e^{-rs} O(T_i, s) ds \\ & + e^{-r(T_{i+1} - T_t)} \Big(- p(T_{i+1}) I_{i+1} + \theta p(T_{i+1}) \frac{e^{-\delta R_i} K_i}{q(T_i)} \Big) \Bigg). \end{aligned}$$

In Appendix F it is shown that using this, the Bellman equation of the problem can be derived and it becomes,

$$\begin{aligned} & V\big(T_t, K_t\big) = \max_{R_t, I_{t+1}} \int_0^{R_t} e^{-rs} O(T_t, s) ds \\ & + e^{-rR_t} \Big(- p(T_{t+1}) I_{t+1} + \theta p(T_{t+1}) \frac{e^{-\delta R_t} K_t}{q(T_t)} + V(T_{t+1}, K_{t+1}) \Big). \end{aligned} \tag{15}$$

At this point it is possible to stop and use Eq. (15) in the second step 4.2.2. However, the analysis is simplified by assuming Cobb-Douglas production technology in line with Mukoyama (cf. assumption (iii)),

(v). $O(T_t, s) = Ak(T_t, s)^\alpha$.

Using this, a simple integration give us the Bellman in which there are only two exogenous functions p and q,

$$V\big(T_t, K_t\big) = \max_{R_t, I_{t+1}} AK_t^\alpha \frac{1 - e^{-(r+\delta\alpha)R_t}}{r + \delta\alpha} \tag{16}$$
$$+ e^{-rR_t}\Big(- p(T_{t+1})I_{t+1} + \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)} + V(T_{t+1}, K_{t+1})\Big).$$

4.2.3 Step 3: Derive Euler equations

The Bellman equation is exploited in several ways in order to fully characterize the optimal paths of control and state variables. From the envelope theorem one gets preliminary result for the optimal paths of state variables. The envelope theorem[10] applied to the state variable T_t yields[11],

$$\frac{\partial}{\partial T_t}V(T_t, K_t) = e^{-rR_t}\Big(- p'(T_{t+1})I_{t+1} + \theta p'(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)} \tag{17}$$
$$- \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)^2}q'(T_t) + \frac{\partial}{\partial T_{t+1}}V(T_{t+1}, K_{t+1}) + \frac{\partial}{\partial K_{t+1}}V(T_{t+1}, K_{t+1})q'(T_{t+1})I_{t+1}\Big).$$

Note that the dynamics of state variables, meaning Eqs. (13) and (12), are utilized when calculating partial derivatives $\frac{\partial}{\partial T_{t+1}}V(T_{t+1}, K_{t+1})$ and $\frac{\partial}{\partial K_{t+1}}V(T_{t+1}, K_{t+1})$. Correspondingly, the envelope theorem applied to the state variable K_t yields,

$$\frac{\partial}{\partial K_t}V(T_t, K_t) = \alpha AK_t^{\alpha-1}\frac{1 - e^{-(r+\delta\alpha)R_t}}{r + \delta\alpha} + \theta e^{-rR_t}p(T_{t+1})\frac{e^{-\delta R_t}}{q(T_t)}. \tag{18}$$

The equations (17) and (18) are costate equations. The next goal is to derive Euler equations for both control variables R_t and I_t. This can be done by using the first-order conditions. Let us first deal with simpler case.

Euler equation of $\mathbf{I_t}$ The necessary condition for (interior) solution of the problem is that partial derivative of control variable I_{t+1} must vanish,

$$- e^{-rR_t}p(T_{t+1}) + e^{-rR_t}\frac{\partial}{\partial K_{t+1}}V(T_{t+1}, K_{t+1})q(T_{t+1}) = 0$$
$$\Leftrightarrow \quad \frac{\partial}{\partial K_{t+1}}V(T_{t+1}, K_{t+1}) = \frac{p(T_{t+1})}{q(T_{t+1})}. \tag{19}$$

[10] Milgrom and Segal showed that the differentiability of value function and objective function in parameters is only required (Milgrom & Segal, 2002, [23]). In order to study the differentiability, some restrictions must be imposed on functions $p(t)$ and $q(t)$. To avoid these questions, the differentiability is just assumed. In light of Proposition 1 one has to only show that if $f(t)$ denotes the objective function, then $\sum_{n=1}^{\infty} f'(t)$ converges uniformly on $[0, \infty[$ when assumptions (vii) and (vi) hold.

[11] Note that due to the envelope theorem, we must not take into account the partial derivatives $\frac{\partial R_t}{\partial K_t}, \frac{\partial R_t}{\partial T_t}, \frac{\partial I_{t+1}}{\partial K_t}$ and $\frac{\partial I_{t+1}}{\partial T_t}$.

By lagging this equation by one period and inserting the result into Eq. (18) gives the evolution of state variable K_t,

$$\frac{p(T_t)}{q(T_t)} = \alpha A K_t^{\alpha-1}\frac{1-e^{-(r+\delta\alpha)R_t}}{r+\delta\alpha} + \theta e^{-rR_t} p(T_{t+1})\frac{e^{-\delta R_t}}{q(T_t)}$$
$$\Leftrightarrow \quad K_t = \left(\frac{\frac{p(T_t)}{q(T_t)} - \theta e^{-rR_t} p(T_{t+1})\frac{e^{-\delta R_t}}{q(T_t)}}{\alpha A \frac{1-e^{-(r+\delta\alpha)R_t}}{r+\delta\alpha}}\right)^{\frac{1}{\alpha-1}}. \tag{20}$$

Using the dynamics of capital stock , it is immediate that the Euler equation of control variable I_{t+1} can be written as,

$$I_{t+1} = \frac{1}{q(T_{t+1})}\left(\frac{\frac{p(T_{t+1})}{q(T_{t+1})} - \theta e^{-rR_{t+1}} p(T_{t+2})\frac{e^{-\delta R_{t+1}}}{q(T_{t+1})}}{\alpha A \frac{1-e^{-(r+\delta\alpha)R_{t+1}}}{r+\delta\alpha}}\right)^{\frac{1}{\alpha-1}}. \tag{21}$$

Euler equation of $\mathbf{R_t}$ Then, Euler equation for the control variable R_t will be derived. The idea is the following. From the first-order condition for R_t, the partial derivatives $\frac{\partial}{\partial T_t}V(T_t, K_t)$ and $\frac{\partial}{\partial T_{t+1}}V(T_{t+1}, K_{t+1})$ can be solved. Besides, the partial derivative $\frac{\partial}{\partial K_{t+1}}V(T_{t+1}, K_{t+1})$ is already solved. Thus, we can get rid of partial derivatives in the costate equation. There will remain some terms that contain $V(T_t, K_t)$ and $V(T_{t+1}, K_{t+1})$. Those can be disposed by using Bellman equation. The derivation is given in full detail in Appendix G, but here it is satisfied with an undetailed derivation.

The first-order condition for control variable R_t reads as,

$$\begin{aligned}
&A K_t^{\alpha} e^{-(r+\delta\alpha)R_t} - re^{-rR_t}\Big(- p(T_{t+1})I_{t+1} + \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)} + V(T_{t+1}, K_{t+1})\Big) \\
&+ e^{-rR_t}\Big(- p'(T_{t+1})I_{t+1} + \theta p'(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)} - \delta\theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)} \\
&+ \frac{\partial}{\partial T_{t+1}}V(T_{t+1}, K_{t+1}) + \frac{\partial}{\partial K_{t+1}}V(T_{t+1}, K_{t+1})q'(T_{t+1})I_{t+1}\Big) = 0.
\end{aligned}$$

From this, the partial derivatives $\frac{\partial}{\partial T_t}V(T_t, K_t)$ and $\frac{\partial}{\partial T_{t+1}}V(T_{t+1}, K_{t+1})$ can be solved. Inserting these and $\frac{\partial}{\partial K_{t+1}}V(T_{t+1}, K_{t+1})$ into the costate equation, we obtain after couple of simplifications,

$$\begin{aligned}
&- A K_{t-1}^{\alpha} e^{-\delta\alpha R_{t-1}} + r\Big(- p(T_t)I_t + \theta p(T_t)\frac{e^{-\delta R_{t-1}}K_{t-1}}{q(T_{t-1})} + V(T_t, K_t)\Big) \\
&+ p'(T_t)I_t - \theta p'(T_t)\frac{e^{-\delta R_{t-1}}K_{t-1}}{q(T_{t-1})} + \delta\theta p(T_t)\frac{e^{-\delta R_{t-1}}K_{t-1}}{q(T_{t-1})} - \frac{p(T_t)}{q(T_t)}q'(T_t)I_t \\
&= e^{-rR_t}\Bigg(- \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)^2}q'(T_t) - A K_t^{\alpha} e^{-\delta\alpha R_t} + \delta\theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)} \\
&+ r\Big(- p(T_{t+1})I_{t+1} + \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)} + V(T_{t+1}, K_{t+1})\Big)\Bigg).
\end{aligned}$$

To get rid of the value functions appearing in the equation, the maximized Bellman equation should be exploited. The maximized Bellman equation is the original Bellman equation (16) given that controls are chosen optimally (in that case the maximal operator disappears). After using this and doing several simplifications, we achieve our goal. The Euler equation of control variable R_t is characterized by the root (roots) of the function G,

$$G(T_t, R_t, R_{t+1}) \stackrel{\text{def}}{=} r\Bigg(AK_{t+1}^{\alpha}\frac{1-e^{-(r+\delta\alpha)R_{t+1}}}{r+\delta\alpha} - p(T_{t+1})I_{t+1} + \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)}\Bigg) \tag{22}$$
$$- AK_t^{\alpha}e^{-\delta\alpha R_t} + p'(T_{t+1})I_{t+1} - \theta p'(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)} + \delta\theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)}$$
$$- \frac{p(T_{t+1})}{q(T_{t+1})}q'(T_{t+1})I_{t+1} - e^{-rR_{t+1}}\Bigg(- \theta p(T_{t+2})\frac{e^{-\delta R_{t+1}}K_{t+1}}{q(T_{t+1})^2}q'(T_{t+1})$$
$$- AK_{t+1}^{\alpha}e^{-\delta\alpha R_{t+1}} + \delta\theta p(T_{t+2})\frac{e^{-\delta R_{t+1}}K_{t+1}}{q(T_{t+1})}\Bigg) \;=\; 0.$$

This is a first-order non-linear difference equation in variable R_t. Even if p and q have a simple form, the existence of analytical solution is very unlikely. Three initial values T_0, K_0 and R_0 have to be first determined. The initial value R_0 is needed since the difference equation is of the first-order. Unfortunately, there is no obvious candidate for that. The initial value can be obtained, for example, by solving a system[12] of first N recursive relations and approximating the last replacement interval by $R_{t+N+1} \approx R_{t+N}$. Given that R_0, T_0 and K_0 are known, an optimal replacement policy can be solved recursively by solving the positive root of the function G for variable R_{t+1} at all times T_t. Recall that from the state equation of variable T_t, Eq. (13), the variable T_t can be derived by induction,

$$T_t = T_0 + \sum_{i=0}^{t-1} R_i, \quad for\ t \in \{1, 2, 3, ...\} \quad and \quad T_0\ is\ given.$$

The determination of initial values T_0 and K_0 is not so problematic. A natural choice is to set $T_0 = 0$ whereas the value of K_0 should be computed as noted in the subsection 4.2.1. Meaning that the initial capital stock must be at the optimal level in the sense of Eq. (20). So, K_0 can be calculated if R_0 and T_0 are known.

4.3 Investment-specific technological change: Two approaches

The model incorporates two approaches to model investments-specific technological change. Therefore, it provides a framework in which the comparison among these approaches is

[12] In order to calculate N optimal replacement intervals, it amounts to solve a system of N equations:

$$\begin{cases} G(T_t, R_t, R_{t+1}) = 0 \\ G(T_{t+1}, R_{t+1}, R_{t+2}) = 0 \\ . \\ . \\ G(T_{t+N-1}, R_{t+N-1}, R_{t+N}) = 0 \end{cases}$$

The problem of an unknown initial value R_0 can be avoided by adding extra equations into the system and then approximating the last replacement interval by $R_{t+x+1} \approx R_{t+x}$.

possible. Particularly, the comparison can be done with respect to the effect they have on the replacement decision and depreciation. The actual comparison is conducted in Section 5.

Two closely related approaches to describe investment-specific technological change can be found from the literature. The first approach describes investments-specific technological change in terms of prices. The investment-specific technological progress can be described as a fall in the price of capital (e.g. Greenwood et al., 1997, [5]; Mukoyama, 2008, [2]; Hulten, 1992, p.967, [24]). The second approach describes investment-specific technological progress as a growth in the relative productivity of new capital (e.g. Deli,2016,p.323,[18]; Greenwood et al., 1997, [5]; Justiano et al., 2010, [25]). Greenwood et al. (1997) and Hulten (1992) both give a comprehensive discussion about how these two descriptions of investment-specific technological change are related to each other. Greenwood et al. show in their framework how the price and the productivity of new capital are inversely related (Greenwood et al., 1997, p.361, [5]). Several authors have made similar observation (e.g. Hulten, 1992, p.967, [24]; Bakhshi & Larsen, 2001, p.15, [26]).

Inspired by these two interpretations of investment-specific technological change, it will be studied how they are related in this framework. Specifically, how they are related with respect to the effect they have on the replacement decision and depreciation. The model incorporates counterparts of these two. The quantity $p(T_t)$ describes the unit price of capital (measured in one unit of output) at time T_t. On the other hand, $q(T_t)I_t$ describes an (efficient) amount of just installed capital stock at time T_t. Hence $q(T_t)$ can be interpreted as the productivity of a new (vintage) capital at time T_t, or as Hulten put it: "*the index $\Phi(t)$ [in our context $q(t)$] can be interpreted as the best-practice level of technology in year t, and the change in $\Phi(t)$ can be interpreted as the quality differential between successive vintages*" (Hulten, 1992, p.966, [24]). Let us focus on the case in which the productivity grows and the price falls, both at a constant rate. That is, q and p are assumed to follow exponential functions with parameters λ and γ respectively

(vi). $q(t) = e^{\lambda t}, \quad where \quad \lambda \geq 0$

(vii). $p(t) = e^{\gamma t}, \quad where \quad \gamma \leq 0$

Parameters λ and γ can be interpreted as annual growth rates of the productivity of new capital and the price of capital[13]. Note that the assumptions (vii) is same as Mukoyama's assumption (viii) up to the sign of parameter γ. The rest of the thesis it is assumed that assumptions (vi) and (vii) hold, unless stated otherwise.

4.4 The existence of a solution and the second-order condition

The existence of a solution and the second-order condition are briefly examined. The concluding answer to the existence is given in Proposition 3 in the next section.

[13] Recall that $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$. This corresponds to an annual growth rate of x when growth rates are compounded continuously.

4.4.1 The existence

The convergence of the series is necessary for the existence question to be even reasonable. The following proposition guarantees the convergence of the series in the case $\theta = 0$.

Proposition 1. *If $\theta = 0$, $\delta > 0$ and $r > (\gamma - \lambda)\frac{\alpha}{\alpha-1}$, then the present value of the plant is finite for all*[14]*replacement and investment policies, $(R_i, I_{i+1})_{i=t}^{\infty}$.*

Proof. It is enough to show that the claim holds for an optimal replacement and investment policy. Namely, the series given an optimal replacement and investment policy is a upper bound for all other series. That is, for $\theta = 0$ we show that the series (14) converges when $r > (\gamma - \lambda)\frac{\alpha}{\alpha-1}$ and sequences $(R_i)_{i=t}^{\infty}$ are $(I_i)_{i=t+1}^{\infty}$ optimally chosen. Let assume that $\theta = 0$. Then i-th term of the series is,

$$e^{-r(T_i - T_t)} \int_0^{R_i} e^{-rs} O(T_i, s) ds - e^{-r(T_{i+1} - T_i)} p(T_{i+1}) I_{i+1}.$$

By using assumptions (v) - (vii), the optimality of investments (21), and the optimality of capital stock (20), the i-th term becomes,

$$\frac{1}{r + \alpha\delta} A e^{r(T_t - T_i)} \left(1 - e^{-(r+\alpha\delta)R_i}\right) \left(\frac{(r+\alpha\delta)e^{(\gamma-\lambda)T_i}}{\alpha A \left(1 - e^{-(r+\alpha\delta)R_i}\right)}\right)^{\frac{\alpha}{\alpha-1}}$$
$$- e^{(\gamma - r - \lambda)R_i + (\gamma - r - \lambda)T_i + rT_t} \left(\frac{(r+\alpha\delta)e^{(\gamma-\lambda)(T_i+R_i)}}{\alpha A \left(1 - e^{-(r+\alpha\delta)R_{i+1}}\right)}\right)^{\frac{1}{\alpha-1}}.$$

The first term of this can be represented as a factor,

$$C_i e^{-rT_i} \left(e^{(\gamma-\lambda)T_i}\right)^{\frac{\alpha}{\alpha-1}} = C_i e^{\left((\gamma-\lambda)\frac{\alpha}{\alpha-1} - r\right)T_i}.$$

Similarly, the second term can be rewritten as,

$$D_i e^{(\gamma-\lambda-r)T_i} \left(e^{(\gamma-\lambda)T_i}\right)^{\frac{1}{\alpha-1}} = D_i e^{\left(\frac{(\alpha-1)(\gamma-\lambda)}{\alpha-1} + \frac{\gamma-\lambda}{\alpha-1} - r\right)T_i} = D_i e^{\left((\gamma-\lambda)\frac{\alpha}{\alpha-1} - r\right)T_i}.$$

Here, the factors C_i and D_i depend only on constants and replacement interval R_i (note that T_t is a given initial time),

$$C_i \stackrel{\text{def}}{=} \frac{1}{r+\alpha\delta} A e^{rT_t} \left(1 - e^{-(r+\alpha\delta)R_i}\right) \left(\frac{r+\alpha\delta}{\alpha A \left(1 - e^{-(r+\alpha\delta)R_i}\right)}\right)^{\frac{\alpha}{\alpha-1}},$$

$$D_i \stackrel{\text{def}}{=} e^{(\gamma-r-\lambda)R_i + rT_t} \left(\frac{(r+\alpha\delta)e^{(\gamma-\lambda)R_i}}{\alpha A \left(1 - e^{-(r+\alpha\delta)R_{i+1}}\right)}\right)^{\frac{1}{\alpha-1}}.$$

To wrap up, i-th term of the series can be presented as,

$$e^{\left((\gamma-\lambda)\frac{\alpha}{\alpha-1} - r\right)T_i}\left(C_i - D_i\right).$$

[14] A replacement policy that leads to negative plan present value is treated as an inadmissible policy. Inadmissible policies are excluded.

Now, consider a ratio,

$$\frac{e^{\left((\gamma-\lambda)\frac{\alpha}{\alpha-1}-r\right)T_i}\left(C_i-D_i\right)}{e^{\left((\gamma-\lambda)\frac{\alpha}{\alpha-1}-r\right)T_i}}=C_i-D_i.$$

Since $\delta>0$ we can assume that the optimal $(R_i)_{i=t}^{\infty}$ is bounded[15]. Since $R_i>0$ for all i, together these imply that either $(R_i)_{i=t}^{\infty}$ oscillates between finite bounds or $(R_i)_{i=t}^{\infty}$ has a finite limit (e.g. in the case $(R_i)_{i=t}^{\infty}$ is monotonic). It easy to see (C_i-D_i can be regarded as a bounded function of R_i and R_{i+1}) that then the ratio either oscillates between finite bounds or it has a finite limit. In any case we have,

$$\limsup_{i\to\infty}\Big(C_i-D_i\Big)<\infty.$$

On the other hand, by the assumption $r>(\gamma-\lambda)\frac{\alpha}{\alpha-1}\ \Leftrightarrow(\gamma-\lambda)\frac{\alpha}{\alpha-1}-r<0$ we know that the following series converges,

$$\sum_{i=t}^{\infty}e^{\left((\gamma-\lambda)\frac{\alpha}{\alpha-1}-r\right)T_i}\ <\ \infty$$

since[16] $T_i=T_t+\sum_{j=t}^{i}R_j\longrightarrow\infty$ as $i\to\infty$. By the one-sided version of the limit comparison test[17], the original series (14) converges. □

The condition $\delta>0$ is not necessary. However, if $\lambda=0=\gamma$, then $\delta>0$ is necessary for the existence. The reason is that if there were neither investment-specific technological change nor physical depreciation, then there would not be any incentive to replace capital stock at all. On the other hand, the condition $r>(\gamma-\lambda)\frac{\alpha}{\alpha-1}$ is a necessary condition for the existence. In fact, along with $\delta>0,\ \lambda>$ or $\gamma<0$, it also constitute a sufficient condition for the existence. This is show in Proposition 3. The condition $r>(\gamma-\lambda)\frac{\alpha}{\alpha-1}$ may seem a bit peculiar at first sight, but recall this is due to the exclusion of the budget constraint. The condition ensures that the discounted profits do not grow too rapidly, because there is available interest-free loan in the financial markets.

4.4.2 The second-order condition

To check the sufficient condition for the maximum, the Hessian matrix of the function determined by the right-hand side of BellmanEquation must be studied. It is straightforward to exclude the possibility of a local minimum.

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[15]Otherwise, there is nothing to prove. To get an idea of this, assume that there does not exist $N>0$ such that $|R_i|<N$ for all i. Then it can be found i such that R_i is arbitrary large. Since $\delta>0$, after time T_i the plant output is (virtually) zero due to capital stock is physically worn out. Hence, the present value of plant is finite.

[16]Technically, it is needed that $(R_i)_{i=t}^{\infty}$ decreases at most at rate $\frac{1}{n}$, i.e. $\frac{1}{n}=\mathcal{O}(R_n)$. Since $\theta=0$ this should be the case. Otherwise within a fixed time frame, the scrapping cost would increase without bounds and thus contradicts with the assumption that $(R_i)_{i=t}^{\infty}$ is an optimal.

[17]Let (x_n) and (y_n) be non-negative sequences. If $\limsup\limits_{n\to\infty}\frac{x_n}{y_n}\in[0,\infty[$ and $\sum_{n=t}^{\infty}y_n$ converges, then necessarily $\sum_{n=t}^{\infty}x_n$ converges

Recall that the first derivative with respect to control variable I_{t+1} reads as,

$$-e^{-rR_t}p(T_{t+1}) + e^{-rR_t}\frac{\partial}{\partial K_{t+1}}V(T_{t+1}, K_{t+1})q(T_{t+1}).$$

After differentiating this second time with respect to I_{t+1}, we obtain

$$e^{-rR_t}\frac{\partial}{\partial K_{t+1}\partial I_{t+1}}V(T_{t+1}, K_{t+1})q(T_{t+1}).$$

By using the forwarded version of Eq. (18), we get

$$e^{-rR_t}(\alpha-1)\alpha A K_{t+1}^{\alpha-2}\frac{1-e^{-(r+\delta\alpha)R_{t+1}}}{r+\delta\alpha}q^2(T_{t+1}) \ < \ 0,$$

which is negative since $(\alpha - 1) < 0$ and other terms are positive. Thus one of the diagonal elements of the Hessian is negative, implying that the critical point cannot be a local minimum, hence it is a local maximum or a saddle point (provided that the Hessian is non-singular).

5 Effects of an investment-specific technological change on the replacement decision and depreciation

This section presents some implications of the model on the replacement decision and capital depreciation. The key results are: (i) the stationarity of the replacement policy (Proposition 3), (ii) the equivalence in that whether investment-specific technological progress is described as a fall in the price of capital or as a growth in the relative productivity of new capital (Proposition 4) and (iii) the intensification of capital replacement due to an acceleration in investment-specific technological progress (Proposition 5). Further, two results shown by Mukoyama (2008) are verified in this general setting. First, the replacement decision is independent of an initial time (cf. Proposition 1 in Mukoyama (2008) and Proposition 2). Second, the producer replaces capital more frequently when a fall in the price of capital accelerates (cf. Proposition 3 in Mukoyama (2008) and Proposition 6) (Mukoyama, 2008, [2]).

5.1 The replacement decision

The difference equation G, which characterizes the optimal replacement, is rather complex. By following Mukoyama and assuming that $\theta = 0$, it possible to derive various analytical results concerning the model. This assumption is posed due to analytical convenience and the case $\theta > 0$ is studied numerically in subsection 5.4.

The following proposition guarantees that the optimal replacement policy is independent of an initial time. The result has relevance on its own right, but it primarily serves as a stepping stone in showing other results.

Proposition 2. *Assume that $\theta = 0$. If $(R_i, I_{i+1})_{i=0}^{\infty}$ is an optimal replacement and investment policy for some initial time T_0, then the function G is independent of time T_t.*

Proof. For given T_0, assume that $(R_i)_{i=0}^{\infty}$,$(I_i)_{i=0}^{\infty}$ and $(K_i)_{i=0}^{\infty}$ satisfy Eq. (22), Eq. (21) and Eq. (20), respectively. We show that the function G vanishes in the region $[0,\infty[\times\{R_t\}\times\{R_{t+1}\}$. Specifically, it is shown that a function defined by $F(x) \stackrel{\text{def}}{=} G(x, R_t, R_{t+1})$ vanishes in $[0,\infty[$ at all periods $t \in \{0, 1, 2, ...\}$.

Fix period $t \in \{0, 1, 2, 3, ...\}$ and assume that an initial time $T_0 \geq 0$ is arbitrary. Note that T_t is determined by $T_t = T_0 + \sum_{i=0}^{t} R_i$ for given replacement policy $(R_i)_{i=0}^{\infty}$. First, it holds that $F(T_t) = G(T_t, R_t, R_{t+1}) = 0$, since R_t and R_{t+1} are chosen optimally. We wish to extend this result into positive reals $[0,\infty[$.

Suppose that $\theta = 0$. After substituting the expressions I_t, K_t, q and p, it is possible to directly differentiate the function G with respect to T_t. In Appendix H it is shown that partial derivative $\frac{\partial}{\partial T_t}G$ can be represented in terms of the function G and parameter values α, γ and λ,

$$\frac{\partial}{\partial T_t}G = (\gamma - \lambda)\frac{\alpha}{\alpha - 1}G. \tag{23}$$

Note that this means $F'(x) = cF(x)$, where $c \stackrel{\text{def}}{=} (\gamma - \lambda)\frac{\alpha}{\alpha-1}$, since a straightforward computation gives,

$$F'(x) = \frac{\partial}{\partial x}G(x, R_t, R_{t+1}) = (\frac{\partial}{\partial T_t}G)(x, R_t, R_{t+1})$$
$$= (\gamma - \lambda)\frac{\alpha}{\alpha - 1}G(x, R_t, R_{t+1}) = (\gamma - \lambda)\frac{\alpha}{\alpha - 1}F(x).$$

Then, assume that $x \geq T_t$ and deduce,

$$F'(x) = cF(x) \quad \Rightarrow \quad e^{-cx}F'(x) - ce^{-cx}F(x) = 0 \quad \Rightarrow \quad D_x\Big(e^{-cx}F(x)\Big) = 0$$
$$\Rightarrow \quad \int_{T_t}^{x} D_s\Big(e^{-cs}F(s)\Big)ds = 0 \quad \Rightarrow \quad e^{-cx}F(x) = e^{-cT_t}F(T_t).$$

Since $F(T_t) = 0$, it holds that $F(x) = 0$ for all $x \geq T_t$. In the case $0 \leq x < T_t$ just note that we have $-\int_{T_t}^{x} D_s\Big(e^{-cs}F(s)\Big)ds = 0$ and similarly deduce that $F(x) = 0$. Thus we conclude that $F(x) = 0$ for all $x \in [0,\infty[$. □

Proposition 2 (in our context) is essentially equivalent to Proposition 1 that appears in Mukoyama (Mukoyama, 2008, p.516, [2]). From Proposition 2, it follows that the stationary replacement policy is a solution to the replacement problem, when a particular transcendental function has a root.

Lemma 1. *Assume that $\theta = 0$. There exists a replacement interval R such that a constant sequence $R_t = R$ for all t is an optimal replacement policy, if and only if, R is a root of the transcendental function H,*

$$H(R) \stackrel{\text{def}}{=} e^{-(\alpha\delta + (\gamma-\lambda)\frac{\alpha}{\alpha-1})R} - \frac{\alpha(r + \delta + \lambda - \gamma)}{r + \alpha\delta}e^{-(r+\alpha\delta)R} - \frac{r - \alpha(r + \lambda - \gamma)}{r + \alpha\delta}. \tag{24}$$

Proof. The optimal replacement policy is characterized by the roots of function G. Proposition 2 implies that functions $G(T_t, R_t, R_{t+1})$ and $G(0, R_t, R_{t+1})$ have identical roots in variables R_t and R_{t+1}. Thus, for the existence of stationary solution, it has to be the case that there exists $R > 0$ such that $G(0, R, R) = 0$. By substituting investments I_t, capital stocks K_t, investment-specific technological change q and capital prices p, it can be shown that this is equivalent to (see Appendix I),

$$Ae^{((\gamma-\lambda)\frac{\alpha}{\alpha-1})R}\left(e^{-(\alpha\delta+(\gamma-\lambda)\frac{\alpha}{\alpha-1})R} - \frac{\alpha(r+\delta+\lambda-\gamma)}{r+\alpha\delta}e^{-(r+\alpha\delta)R} - \frac{r-\alpha(r+\lambda-\gamma)}{r+\alpha\delta}\right) = 0.$$

□

Lemma 1 plays the crucial role in the forthcoming analysis. This due to the fact that the function H fully characterizes the stationary replacement policy. On the other hand, Lemma 1 has a negative implication. It is very unlikely that there exists a closed-form solution for variable R, because there does not exist a general method for solving this type of transcendental equation. The existence and the uniqueness of the stationary replacement policy can be shown by studying the function H. This is the content of Proposition 3.

Proposition 3. *Assume that $\theta = 0$, $r > (\gamma-\lambda)\frac{\alpha}{\alpha-1}$ and either one holds $\delta > 0$, $\lambda >$ or $\gamma < 0$. Then, there exists a unique replacement interval R such that a constant sequence $R_t = R$ for all t is an optimal replacement policy.*

Proof. Suppose that $\delta > 0$, $\lambda >$ or $\gamma < 0$, then $\alpha\delta + (\gamma-\lambda)\frac{\alpha}{\alpha-1} > 0$. Further, assume that $r > (\gamma-\lambda)\frac{\alpha}{\alpha-1}$. Together these justify the following calculation,

$$\frac{\partial H}{\partial R} = \alpha(r+\delta+\lambda-\gamma)e^{-(r+\alpha\delta)R} - (\alpha\delta + (\gamma-\lambda)\frac{\alpha}{\alpha-1})e^{-(\alpha\delta+(\gamma-\lambda)\frac{\alpha}{\alpha-1})R} < 0$$

$$\Leftrightarrow \quad R > \frac{\log\frac{\alpha(r+\delta+\lambda-\gamma)}{\alpha\delta+(\gamma-\lambda)\frac{\alpha}{\alpha-1}}}{r-(\gamma-\lambda)\frac{\alpha}{\alpha-1}} \overset{\text{def}}{=} R^*.$$

Hence, the function H is strictly decreasing on $]R^*, \infty[$ and strictly increasing on $]0, R^*[$. On the other hand, we have $H(R^*) > 0$ since $H(R^*) > H(0) = 1 - \frac{\alpha(r+\delta+\lambda-\gamma)}{r+\alpha\delta} - \frac{r+\alpha(\gamma-\lambda-r)}{r+\alpha\delta} = 0$. Since $\lim_{R\to\infty} H(R) = -\frac{r+\alpha(\gamma-\lambda-r)}{r+\alpha\delta} < 0$, for large enough R' it holds that $H(R') < 0$. By the Bolzano's theorem, there exits $R \in]R^*, \infty[$ such that $H(R) = 0$. Furthermore, the root is unique since H is monotonic on $]R^*, \infty[$ and $R = 0$ is not valid root because it is not an applicable replacement policy. □

To guarantee the time-independence of the replacement policy, is no longer needed a kind of mechanical assumption (ii) that was imposed on a functional form of capital stock in Mukoyama's model (2008). The time independence of the replacement policy is an inherent result of the model due to increasing investments (21) in the presence of an investment-specific technological growth. That is, the producer reacts to *a steady growth* of investment-specific technology by accommodating investments alone, not replacement policy. The situation is totally different when there is going on a *change in growth rate* of investment-specific technological change. In that case, as soon it will turn out (see Proposition 6), the producer also accommodates replacement policy.

There is a need for the bounds for variable R due to the lack of a closed-form solution. The following lemma has a technical advantage when conducting comparative statics, as well, it gives a reference point on constructing approximation for variable R.

Lemma 2. *Assume that $\theta = 0$, $r > (\gamma - \lambda)\frac{\alpha}{\alpha-1}$ and either one holds $\delta > 0$, $\lambda >$ or $\gamma < 0$. Then, the optimal replacement policy satisfies* $-\frac{\log \frac{r-\alpha(r+\lambda-\gamma)}{r-(\gamma-\lambda)\frac{\alpha}{\alpha-1}}}{(\alpha\delta+(\gamma-\lambda)\frac{\alpha}{\alpha-1})} < R < -\frac{\log \frac{r-\alpha(r+\lambda-\gamma)}{r+\alpha\delta}}{(\alpha\delta+(\gamma-\lambda)\frac{\alpha}{\alpha-1})}$.

Proof. From the optimality condition $H(R) = 0$ we get,

$$\alpha(r+\delta+\lambda-\gamma)e^{-(r+\alpha\delta)R} = (r+\alpha\delta)e^{-(\alpha\delta+(\gamma-\lambda)\frac{\alpha}{\alpha-1})R} - r - \alpha(\gamma-\lambda-r), \tag{25}$$

which implies that $\frac{\partial H}{\partial R}$ takes a form (substitute Eq. (25) into $\frac{\partial H}{\partial R}$),

$$\begin{aligned}
&(r+\alpha\delta)e^{-(\alpha\delta+(\gamma-\lambda)\frac{\alpha}{\alpha-1})R} - r - \alpha(\gamma-\lambda-r) - (\alpha\delta+(\gamma-\lambda)\frac{\alpha}{\alpha-1})e^{-(\alpha\delta+(\gamma-\lambda)\frac{\alpha}{\alpha-1})R} \\
&= \big(r-(\gamma-\lambda)\frac{\alpha}{\alpha-1}\big)e^{-(\alpha\delta+(\gamma-\lambda)\frac{\alpha}{\alpha-1})R} - r - \alpha(\gamma-\lambda-r)
\end{aligned}$$

On the other hand, as shown in Proposition 3, it must hold $\frac{\partial H}{\partial R} < 0$ at the optimal R. That is, at $\frac{\partial H}{\partial R}\big|_{R \ s.t \ H(R)=0}$ it holds,

$$\begin{aligned}
&\frac{\partial H}{\partial R} \ < \ 0 \ \Leftrightarrow \log\Big(r-(\gamma-\lambda)\frac{\alpha}{\alpha-1}\Big) - (\alpha\delta+(\gamma-\lambda)\frac{\alpha}{\alpha-1})R \ < \ \log\Big(r-\alpha(r+\lambda-\gamma)\Big) \\
&\Leftrightarrow R \ > \ \frac{1}{-(\alpha\delta+(\gamma-\lambda)\frac{\alpha}{\alpha-1})}\log\left(\frac{r-\alpha(r+\lambda-\gamma)}{r-(\gamma-\lambda)\frac{\alpha}{\alpha-1}}\right).
\end{aligned}$$

The upper bound can be derived from $H(R) = 0$. Then,

$$\begin{aligned}
&e^{-(\alpha\delta+(\gamma-\lambda)\frac{\alpha}{\alpha-1})R} = \frac{\alpha(r+\delta+\lambda-\gamma)}{r+\alpha\delta}e^{-(r+\alpha\delta)R} + \frac{r-\alpha(r+\lambda-\gamma)}{r+\alpha\delta} \\
&\Leftrightarrow \ e^{(r-(\gamma-\lambda)\frac{\alpha}{\alpha-1})R} = \frac{\alpha(r+\delta+\lambda-\gamma)}{r+\alpha\delta} + \frac{r-\alpha(r+\lambda-\gamma)}{r+\alpha\delta}e^{(r+\alpha\delta)R}
\end{aligned}$$

Use the identity $\log(x+y) = \log(x) + \log(1+\frac{y}{x})$,

$$(r-(\gamma-\lambda)\frac{\alpha}{\alpha-1})R = \log\Big(\frac{r-\alpha(r+\lambda-\gamma)}{r+\alpha\delta}e^{(r+\alpha\delta)R}\Big) + \log\Big(1+\frac{\alpha(r+\delta+\lambda-\gamma)}{r-\alpha(r+\lambda-\gamma)}e^{-(r+\alpha\delta)R}\Big)$$

The condition $r > (\gamma-\lambda)\frac{\alpha}{\alpha-1}$ implies $r > \alpha(r+\lambda-\gamma)$. Hence,

$$\begin{aligned}
&\log\Big(1+\frac{\alpha(r+\delta+\lambda-\gamma)}{r-\alpha(r+\lambda-\gamma)}e^{-(r+\alpha\delta)R}\Big) \ > \ 0 \\
&\Rightarrow \ -(\alpha\delta+(\gamma-\lambda)\frac{\alpha}{\alpha-1})R \ > \ \log\Big(\frac{r-\alpha(r+\lambda-\gamma)}{r+\alpha\delta}\Big) \\
&\Leftrightarrow \ R \ < \ -\frac{\log\left(\frac{r-\alpha(r+\lambda-\gamma)}{r+\alpha\delta}\right)}{(\alpha\delta+(\gamma-\lambda)\frac{\alpha}{\alpha-1})}
\end{aligned}$$

□

Even the first iteration of Newton's method with the initial guess that equals to the upper bound in Lemma 2, would provide a quite accurate estimate for variable R. Alternatively, for nicer expression but not so accurate, the average of bounds can be taken as a rough estimate for variable R (when $\theta = 0$),

$$-\frac{\log \frac{(r-\alpha(r+\lambda-\gamma))^2}{(r+\alpha\delta)\left(r-(\gamma-\lambda)\frac{\alpha}{\alpha-1}\right)}}{2\left(\alpha\delta+(\gamma-\lambda)\frac{\alpha}{\alpha-1}\right)}. \tag{26}$$

5.2 Effects of I-S technology on the replacement decision

Thus far, a few desirable features of the model has been established. From this ground, it can be proceeded to study how a change in the growth rate of the price of capital or a change in the growth rate of productivity of new capital would affect on the replacement decision. How these two approaches of modeling investment-specific technological change are related? When scrapped capital stock has no value, then the answer is unambiguous.

Proposition 4. *Suppose that scrapped capital stock has no value, that is $\theta = 0$. Then, an one unit increase in λ or an one unit decrease in γ have identical effect on the optimal replacement interval R.*

Proof. By looking the function H in Lemma 1, it can be seen that parameters λ and $-\gamma$ are in symmetric relation in the sense that if one denotes $a = \gamma - \lambda$, then H is not a function of λ or $-\gamma$, but only a function of a. □

Greenwood et al. (1997) writes "*...movements in q can be interpreted in two different ways. First, $1/q$ could be thought of as representing the cost of producing a new unit of equipment [in contrast to structure type of capital] in terms of final output...*" (Greenwood et al., 1997, [5]). When the productivity and the price of capital evolves according (vi) and (vii), then Proposition 4 gives a formal proof to support this interpretation in our context. To see this, note that $1/q(t) = e^{-\lambda t}$ and $p(t) = e^{\gamma t}$. By Proposition 4, $\partial R/\partial \gamma = -\partial R/\partial \lambda = \partial R/\partial(-\lambda)$, thus $1/q = p$ with respect to the effect they have on the replacement interval. Nevertheless, this not the whole truth. If one assumes that scrapped capital stock has some value, that is $\theta > 0$, then it turns out that the the productivity and the price change in capital goods do not have identical effect on replacement decision. This observation is made in numerical analysis (see subsection 5.4).

Still, it should be figure out what is an actual effect of I-S technological change on the replacement problem. Mukoyama (2008) showed[18] that the optimal R is increasing in γ when $\theta = 0$ and $(r + \alpha\gamma/(1-\alpha))R > 1$ (Mukoyama, 2008, [2]). Therefore one may anticipate that this kind of result should also hold in more general setting. In fact this is the case, which is shown in Proposition 6. In light of Proposition 4, it can be proceeded by showing first the claim for parameter λ.

Proposition 5. *Assume that $\theta = 0$, $r > (\gamma - \lambda)\frac{\alpha}{\alpha-1}$ and either one holds $\delta > 0$, $\lambda >$ or $\gamma < 0$. Then, the optimal replacement policy R is decreasing in λ, if $R > (1-\alpha)\frac{1-e^{-(r+\alpha\delta)R}}{r-\alpha(r+\lambda-\gamma)}$.*

Proof.

$$\frac{\partial H}{\partial \lambda} = \frac{\alpha}{\alpha - 1} R e^{-(\alpha\delta + (\gamma-\lambda)\frac{\alpha}{\alpha-1})R} - \frac{\alpha}{r+\alpha\delta} e^{-(r+\alpha\delta)R} + \frac{\alpha}{r+\alpha\delta}$$

[18] The precise formulation is: "*Suppose that $\theta = 0$ and $T < \infty$. The optimal T is decreasing in γ, if $(r - \alpha\gamma/(1-\alpha))T > 1$.*" (Mukoyama, 2008, p.518, [2]). In our context T=R and instead of $p(t) = e^{-\gamma t}$ we have $p(t) = e^{\gamma t}$.

By exploiting the upper bound appearing in Lemma 2,

$$\frac{\partial H}{\partial \lambda} < \frac{\alpha}{\alpha-1} R\Big(\frac{r-\alpha(r+\lambda-\gamma)}{r+\alpha\delta}\Big) - \frac{\alpha}{r+\alpha\delta} e^{-(r+\alpha\delta)R} + \frac{\alpha}{r+\alpha\delta}$$
$$= \underbrace{\frac{\alpha e^{-(r+\alpha\delta)R}}{(r+\alpha\delta)(\alpha-1)}}_{\text{-}} \Big(\underbrace{\alpha(e^{(r+\alpha\delta)R}-1)+1+e^{(r+\alpha\delta)R}R(r-\alpha(r+\lambda-\gamma))-e^{(r+\alpha\delta)R}}_{+}\Big)$$

The last term is positive, if and only if, $R > (1-\alpha)\frac{1-e^{-(r+\alpha\delta)R}}{r-\alpha(r+\lambda-\gamma)}$. Note that $r > (\gamma-\lambda)\frac{\alpha}{\alpha-1}$ implies $r-\alpha(r+\lambda-\gamma) > 0$, meaning that the bound is positive.

We have shown that $\frac{\partial H}{\partial \lambda} < 0$. By looking the proof of Proposition 3, it is immediate that $\frac{\partial H}{\partial R} < 0$. From the implicit function theorem,

$$\frac{dR}{d\lambda} = -\frac{\frac{\partial H}{\partial \lambda}}{\frac{\partial H}{\partial R}} \ < \ 0$$

□

Remark. *Alternatively, it can be directly shown from the expression of* $\frac{\partial H}{\partial \lambda}$ *that* $\frac{dR}{d\lambda} < 0 \ \Leftrightarrow \ R > \frac{1-\alpha}{r+\alpha\delta} e^{(\gamma-\lambda)\frac{\alpha}{\alpha-1}R}\big(e^{\alpha\delta R} - e^{-rR}\big)$

Proposition 6. *Assume that* $\theta = 0$, $r > (\gamma-\lambda)\frac{\alpha}{\alpha-1}$ *and either one holds* $\delta > 0, \ \lambda >$ *or* $\gamma < 0$. *Then, the optimal replacement policy* R *is increasing in* γ, *if* $R > (1-\alpha)\frac{1-e^{-(r+\alpha\delta)R}}{r-\alpha(r+\lambda-\gamma)}$.

Proof. The result readily follows from Proposition 4 and Proposition 5. □

Due to the transcendental nature of the function H, it is difficult to derive a nice expression that serves as a sufficient condition or as a necessary condition. The condition $R > (1-\alpha)\frac{1-e^{-(r+\alpha\delta)R}}{r-\alpha(r+\lambda-\gamma)}$ is the sufficient condition, but not the necessary condition. The necessary (and the sufficient) condition $R > \frac{1-\alpha}{r+\alpha\delta} e^{(\gamma-\lambda)\frac{\alpha}{\alpha-1}R}\big(e^{\alpha\delta R} - e^{-rR}\big)$ seems to hold[19] in all reasonable parameter combinations. When $\theta > 0$, it not so clear whether parameters λ and γ have positive or negative effect on R. A quantitative study (see subsection 5.4) supports the view that Proposition 5 and Proposition 6 also hold in that case.

The effects of investment-specific technological change have been examined until now. How does neutral technological change impact on the replacement decision? To answer comprehensively this question, some modifications[20] has to be made to the model. However, it is clear that the *level* of Total Factor Productivity (TFP) does not have an effect on the replacement problem.

Proposition 7. *Assume that* $\theta = 0$. *The optimal replacement policy* R *is independent of* A.

[19]A quantitative inspection shows that the condition seems to hold for all reasonable parameter combinations. For instance, the condition holds for all parameter combinations appearing in subsection 5.4.

[20]For instance, the Cobb-Douglas production assumption must be changed to $O(T_t, s) = A(T_t + s)k(T_t, s)^\alpha$, where $A(t)$ is now a function of time t.

Proof. This is a direct consequence of Lemma 1, since the function H does not depend on the value of A. □

The growth in the productivity of new capital and the decline in the price of capital both generate an opportunity cost when capital replacement is delayed. Delaying the replacement amplifies an incentive to replace capital stock, since there are available gradually more productive capital in the capital markets over time. In the case of declining capital prices, this means that there are available physically in perfect shape capital in the markets perpetually in cheaper prices. On the contrary, the change in the level of TFP does not generate an opportunity cost concerning the replacement decision. The reason is that TFP affects the productivity of the *existing capital* as well. Hence, TFP has no effect on the timing of the replacement. However, from Eq. (21) it can be seen that TFP increases the level of investments.

5.3 Depreciation rate

One way to think how the replacement problem is related to depreciation, is to identify the replacement policy with the retirement distribution in National Accounts. Recall that predominately used method of determining capital stock in National Accounts encompasses two stages in which the life of investments are determined (see Sec.2.2 with Sumit Dey & Chowdhuty, ONS, 2008, [6] and SNA, 2008, p.124, [8]). The first is the determination of investment life length by means of the retirement distribution. The second is the determination depreciation of investment over its lifetime by means of the depreciation function. Replacement interval R_t at time T_t can be identified[21] with (degenerated) retirement distribution of investment at time T_t. In other words, the life-length of capital that is installed at time T_t is R_t. The depreciation function in this case is a geometric depreciation function given by $e^{-\delta t}$. Nevertheless, in our framework at plant level there is no capital accumulation as it appears in Perpetual Inventory Method (i.e. capital stock is a sum of depreciation corrected investments). Conventional methods to calculate depreciation rate may not apply. However, by considering an economy with multiple plants, depreciation rate can be computed "on average". The derivation of a depreciation formula by Mukoyama is drawn upon this idea.

Mukoyama proposes that depreciation and the frequency of capital replacement are related to each other through the formula (6) (Mukoyama, 2008, [2]). In general case, the depreciation rate cannot be calculated by this formula due to two reasons. Firstly, the replacement of capital may not occur at same rate in every period. Mukoyama makes this type of steady-state assumption (Mukoyama, 2008, [2], p.518). Secondly, if the assumption (ii) is relaxed, which concerns the determination of capital stock, then terms $k(\cdot,\cdot)$ have to be replaced somehow. However, a formula that is fitting in more general context can be found by slightly modifying Mukoyama's formula.

5.3.1 Non-stationary depreciation rate

The replacement of capital may not occur at same rate every period. This can be the case whenever the evolution of I-S technological change deviates from those determined

[21] If $\theta = 0$, then the retirement distribution of investment at time T_t can be identified with Dirac delta measure δ on $[T_t, \infty]$ with the property that $\delta(\{T_t + R_t\}) = 1$ and zero otherwise.

in the assumptions (vi) and (vii). For instance, assuming $p(t) = 1$ and $q(t) = e^{\lambda t} + 0.1t$ does the job. This implies that the depreciation rate has to be determined as a function of (continuous) time t. The immediate problem we run into here is that from the plant optimization problem, only a discrete set of optimal choices can be obtained. That is, the replacement and investment policy, $(R_i, I_i, K_i)_{i=0}^{\infty}$, is known at a discrete points in time, $T_0, T_1, T_2, \ldots$ On contrary, continuous paths of $t \mapsto R(t)$, $t \mapsto I(t)$ and $(t, s) \mapsto k(t, s)$ are needed. For instance, consider the integrator $k(t - s, s)$ in Mukoyama's formula (6). To evaluate this, it is required to known the whole path of $(t, s) \mapsto k(t - s, s)$. The underlying reason for this problem is that there is going on a transition from a "microeconomic capital stock" to "macroeconomic capital stock" when trying to derive a formula for depreciation. The formula for depreciation rate introduced by Mukoyama is "economy-wide" (in the sense of the aggregation of homogeneous plants), not "plant specific" such as other quantities derived from the underlying framework are. Therefore, one must deduce from the investment decision $(I_i)_{i=0}^{\infty}$ of a plant, how otherwise identical plants but in different timing cycle decide their optimal investment paths $t \mapsto I(t)$. The path $t \mapsto R(t)$ should be first determined by solving the Euler equation of the variable R_t for all non-negative initial values. That is, one should solve 22 for all non-negative $T_0 \geq 0$. The obtained $\big(R(t)\big)_{t\geq 0}$ extends $(R_i)_{i=0}^{\infty}$ from $\mathbb{N}_0$ onto $[0, \infty]$ such that $R(T_t) = R_t$ for all t. Given the path of the optimal replacement times $\big(R(t)\big)_{t\geq 0}$, the investments can be extended onto positive real line (see Appendix J),

$$I(t) = \frac{1}{q(t)} \left(\frac{\frac{p(t)}{q(t)} - \theta e^{-rR(t)} p(t + R(t)) \frac{e^{-\delta R(t)}}{q(t)}}{\alpha A \frac{1 - e^{-(r+\delta\alpha)R(t)}}{r + \delta\alpha}} \right)^{\frac{1}{\alpha - 1}}. \tag{27}$$

Correspondingly, the capital stock can be continuously determined as,

$$k(t, s) = e^{-\delta s} \left(\frac{\frac{p(t)}{q(t)} - \theta e^{-rR(t)} p(t + R(t)) \frac{e^{-\delta R(t)}}{q(t)}}{\alpha A \frac{1 - e^{-(r+\delta\alpha)R(t)}}{r + \delta\alpha}} \right)^{\frac{1}{\alpha - 1}}.$$

Let us follow Mukoyama by assuming that the economy is in the steady-state in the sense that a number of plants is a constant. Then, the plant age distribution follows the uniform distribution $U(0, R(t))$ at time t. The capital stock terms have to be modified in similar fashion as the term (10) is modified. The terms of the form $k(t - s, s)$ must be replaced by $\frac{k(t-s,s)}{q(t-s)}$. The rationale was given in Appendix D. Further, it may be convenient to switch the timing when depreciation is evaluated, from the beginning of the period $[t, t + R(t)]$ (i.e. t) to the end of the same period (i.e. $t + R(t)$). Taking account these changes in Mukoyama's formula (6), the depreciation rate d at time t can be calculated as (see Appendix K),

$$d(t) = \frac{(1 - \theta) \dfrac{k(t, R(t))}{q(t)}}{\displaystyle\int_0^{R(t)} \frac{k(t + R(t) - s, s)}{q(t + R(t) - s)} ds} + \delta.$$

Since $k(t,s) = e^{-\delta s} q(t) I(t)$, this can be equivalently written in terms of investments,

$$d(t) = \frac{(1-\theta)e^{-\delta R(t)} I(t)}{\displaystyle\int_0^{R(t)} e^{-\delta s} I(t + R(t) - s) ds} + \delta. \tag{28}$$

The determination of the path $\big(R(t)\big)_{t\geq 0}$ is computationally inconvenient due to the fact that it is implicitly defined by the *family* of non-linear difference equations: $G(T_0, R_t, R_{t+1}) = 0$ for all T_0. Therefore, one may be satisfied with an approximation for $R(t)$. From the numerical point of view, a spline interpolation is an easy solution. Even a liner interpolation in every sub-period $u \in [T_t, T_t + R_t]$ gives rather good approximation,

$$R(u) = \frac{R_{t+1} - R_t}{R_t} u + R_t + \frac{R_{t+1} - R_t}{R_t} T_t. \tag{29}$$

5.3.2 Stationary depreciation rate

Let us now assume that the evolution of I-S technological change is described by the assumptions (vi) and (vii). Then by Proposition 3, it is known that $R_t = R$ for all t, when it is further assumed that $\theta = 0$. Let us presume that Proposition 3 holds even if $\theta > 0$ - from the numerical results (see subsection 5.4) this assumption seems to be plausible. What it can be say about $t \mapsto R(t)$? Proposition 2 excludes the possibility $R(u) \neq R_t$ when $u \in [T_t, T_{t+1}]$, since $t \mapsto R(t)$ is characterized by the function G. Therefore, $R(t)$ is a constant R, and the formula (28) takes a form,

$$d(t) = \frac{(1-\theta)e^{-\delta R} I(t)}{\displaystyle\int_0^{R} e^{-\delta s} I(t + R - s) ds} + \delta.$$

Now, the determination of paths $(t,s) \mapsto k(t,s)$ or $t \mapsto I(t)$ is not a problem. Since $R(t) = R$, a new function can be just defined by $I(t) \stackrel{\text{def}}{=} I_{t/R}$ that extends the optimal investment sequence (21) onto reals $[0, \infty[$. Assumption $T_0 = 0$ implies that $T_t = tR$ by Eq. (13). Using these observations, it is shown in Appendix L that the formula simplifies to an expression, which is independent of a time t,

$$d = \frac{(1-\theta)(\delta + \frac{\alpha\lambda - \gamma}{1-\alpha})}{e^{(\delta + \frac{\alpha\lambda-\gamma}{1-\alpha})R} - 1} + \delta. \tag{30}$$

The formula (30) corresponds to Mukoyama's formula of depreciation (6) (Mukoyama, 2008, p.518, [2]). They are equivalent except an inclusion of additional term $\alpha\lambda$ in the formula (30). The appearance of the term $\alpha\lambda$ is due to the fact that investment-specific technological change is incorporated into the model in two ways. After rewriting the formula more concisely,

$$d = \frac{(1-\theta)\phi'}{e^{\phi' R} - 1} + \delta, \quad \textit{where } \phi' \stackrel{\text{def}}{=} \delta + \frac{\alpha\lambda - \gamma}{1-\alpha}, \tag{31}$$

it can be seen that the formulas (31) and (6) are identical up to redefining[22] ϕ as $\phi' = \delta + (\alpha\lambda - \gamma)/(1-\alpha)$. An intriguing fact is that parameter λ is multiplied by α whereas γ

[22]Note that the sign of γ is reversed, because of $p(t) = e^{-\gamma t}$ in the Mukoyama's model.

is not in the formula. Due to an exponential in the denominator, this seems to imply that λ has a greater direct effect on depreciation rate than γ, although this should be verified numerically. The effect of λ on depreciation approaches the effect of $-\gamma$ on depreciation, as α approaches 1. Thus, when $\theta = 0$ and α is close to one, p and $1/q$ have almost identical effect on depreciation rate (recall Proposition 4). The fact, that the effect of q on capital depreciation heavily depends on production technological parameter α, enlightens how q intrinsically reflects technological aspect of capital depreciation whereas this is not only aspect that p reflects.

The interpretation of Mukoyama's (2008) formula (6) also applies to the formula (31). Mukoyama note that *"the first term is determined endogenously"* by the replacement decision (Mukoyama, 2008, p.519, [2]). Conversely, the second term is due to the physical depreciation, and hence it is exogenous with respect to the producer's decision. Endogenous part is negatively related to R, because it always holds that $\phi' > 0$. By Proposition 5 and Proposition 6 this seems to imply that when λ increases or γ decreases, endogenous part increases, consequently d increases. However, d also directly depends on parameter values λ and γ, hence the decisive effect should be verified quantitatively. This is one of the objectives in the next subsection, and it will be also confirmed that d is increasing in λ and decreasing in γ.

Finally, the results are compared to those of Greenwood et al. (Greenwood et al., 1997, [5]). The formula proposed by Greenwood et al., (2), can be represented (after some algebra) as $d = (\Delta q_t/q_t)(1-\delta)+\delta$. Thus, the endogenous part of the formula (6) corresponds to $(\Delta q_t/q_t)(1-\delta)$ when q growths at a steady constant rate (i.e. (vi) holds). If one approximates $(\Delta q_t/q_t) \approx q'(t)/q(t) = \lambda$, then the difference between depreciation rates implied by the formulas equals to,

$$\left|\frac{(1-\theta)(\delta+\frac{\alpha\lambda}{1-\alpha})}{e^{(\delta+\frac{\alpha\lambda}{1-\alpha})R}-1} - \lambda(1-\delta)\right|,$$

which is actually quite small. For instance, when $r = 0.056, \alpha = 1/3, \delta = 0.04, \theta = 0$ and $\lambda = 0.03$, then the difference is only 0.0082, that is, obsolescence is misestimated about 0.8%.

5.4 Numerical analysis

There are still a few unanswered questions. How producer would react, if some of the value of scrapped capital stock was rebated? To answer this, it should be studied what would be the effect of investment-specific technological when parameter θ is strictly positive. The second question is what is the relationship of the generalized model to Mukoyama's (2008) model. How well the results of two models coincide? The third question concerns the relationship of physical depreciation to obsolescence. Which one is more substantial: physical depreciation or obsolescence? The questions are studied quantitatively. Whenever different parameter profiles are considered, one should bear in mind the condition for the convergence posed in Proposition 1 in order to avoid pathological results.

The first concern is that whether the optimal replacement policy is still independent of an initial time and whether it is stationary. This turns out to be the case, after a number of quantitative experiments were conducted by varying parameter profiles. This suggest that Proposition 2 and Proposition 3 may also hold when $\theta > 0$. From now on, all the result concerning the optimal replacement policy can be presented by one number R (which corresponds to T in Mukoyama (2008)) since $R = R_t$ for all t. In light of Proposition 7, an expected results is that TFP would not have effect on the replacement decision even if $\theta > 0$. In fact, this is the case. TFP has only effect on the optimal investment policy and on the value of the plant. This means that there are no "free parameters" that can be used for calibration purposes in contrast[23] to Mukoyama's model. Based on this, it can be argued that the implications of the generalized model are not so model specific.

The parameter values are matched to yearly data such that these are as close as possible to those used by Mukoyama. The parameter A is the only one which can differ due to reason just explained. Since the value of A does not matter, it is set to neural value $A = 1$. The results remain same for all choices $A > 0$. The interest rate and the share of capital in production function are set to $r = 0.056$ and $\alpha = 1/3$ in line with Greenwood and Yorukoglu (Greenwood at al, 1997, [5]; Greenwood & Yorukoglu, 1997, [27]). The rest of parameters, λ and γ concerning investment-specific technological change as well as δ and θ concerning depreciation of capital stock, are varied according the question at hand.

To compare the results with those obtained by Mukoyama, the tables 3 and 4 that appear in Mukoyama (2008) are reproduced in Table 1 and in Table 2, respectively (cf. Mukoyama, 2008, p.520, [2]). In Mukoyama's model the dependence of parameter A along with the fact that it was used to calibration, complicates the comparison. Especially, the levels of replacement interval R and depreciation rate d differ a lot between the models due to the calibration. However, in both models, R and d react similarly to the change in parameters γ, θ and δ. To sum up, as expected from previous considerations (cf. Proposition 1 and Proposition 3 with Mukoyama's (2008) results), the qualitative features of the models coincide, but the levels of variables differ.

Table 1 (*corresponds to Table 3 in Mukoyama (2008)*)
Results with $\alpha = 1/3$, $\delta = 0$

		R (years)	d (%)
$\theta = 0.3$ ($A > 0$)	γ = -0.03	37.3	0.72
	γ = -0.05	24.7	0.98
$\theta = 0.7$ ($A > 0$)	γ = -0.03	29.2	0.50
	γ = -0.05	19.3	0.69

Table 2 (*corresponds to Table 4 in Mukoyama (2008)*)
Results with $\alpha = 1/3$, $\theta = 0.5$

		R (years)	d (%)
$\delta = 0.02$ ($A > 0$)	γ = -0.03	24.7	2.82
	γ = -0.05	18.1	3.04
$\delta = 0.04$ ($A > 0$)	γ = -0.03	19.5	5.00
	γ = -0.05	15.2	5.22

[23] Mukoyama assumes that "*The value of A is picked so that before the change in γ, d is 10%.*" (Mukoyama, 2008, p. 7, [2])

How results would differ if investment-specific technological change was described differently? Instead of assuming that there is a steady fall in the price of capital, that is $\gamma < 0$, let us assume that the productivity of a new unit of capital grows at a steady-rate $\lambda > 0$. In the light of Proposition 4, one may expect that when changing parameters γ or λ, there should be an identical effect on replacement interval R. Surprisingly, this not true when $\theta > 0$. Too see this, let us set parameter values to those in Table 2 with the exception of $\gamma = 0$ and $\lambda > 0$. Table 3 illustrates the effect of λ on replacement interval R and depreciation d. The ratio of obsolescence to depreciation is also reported. When comparing Table 2 and Table 3, it can be observed that parameter λ has a greater effect on the replacement decision than parameter γ. The underlying reason is that when scrapped capital stock has some value, then the change in the price of capital causes two opposing effects for the optimal R as discussed in Mukoyama (Mukoyama, 2008, p.518, [2]). He writes,

> "*First, when γ is large [in our context $-\gamma$ is small], the loss of the old capital's value is faster. Therefore, the marginal cost of waiting is high, and there is an incentive for firm to sell the old capital earlier, before it loses value. Second, the value of the old capital after (given) T [in our context R] time period is smaller. Therefore, the revenue from selling the old capital is less important, and firm is willing to wait longer.*".

These two opposing effects exist due to the term $\theta p(T_{i+1})\frac{e^{-\delta R_i}K_i}{q(T_i)}$ in the objective function. The term can be written as $\theta p(T_{i+1})e^{-\delta R_i}I_i$, when it is easy to note that q is related to this term only through investments I_i. That explains why q and p (hence λ and γ) are not in a symmetric relation with respect to the replacement decision whenever $\theta > 0$. The intuition behind is that there is no aforementioned "second effect" when λ is positive and γ is zero, that is when only the productivity of a new unit of capital grows. After given R time period, the revenue from selling the old capital is not less important.

Table 3
Results with $\alpha = 1/3$, $\theta = 0.5$, $\gamma = 0$

		R (years)	d (%)	$\frac{d-\delta}{d}$ (%)
$\delta = 0.02$ $(A > 0)$	$\lambda = 0.03$	21.3	3.58	44.11
	$\lambda = 0.05$	15.6	4.21	52.52
$\delta = 0.04$ $(A > 0)$	$\lambda = 0.03$	17.5	5.71	29.89
	$\lambda = 0.05$	13.4	6.34	36.92

There also exits an issue concerning the relationship of p with q. Typically, prices are endogenously determined (at least in the general equilibrium context) whereas technology is treated as an exogenous process. Therefore, the price of capital can be viewed as a function of I-S technology, that is $p = p(t, q(t))$. In that case, the situation in which both $\dot{q} > 0$ and $\dot{p} = 0$ hold same time may not be possible. Recall that when $\theta = 0$ and the assumptions (vi) and (vii) hold, then $1/q(t) = p(t)$ in the sense of the effect they have on replacement frequency (see Proposition 4). Therefore, it is interesting study how replacement frequency reacts if the price of capital is determined by I-S technological change, particularly by the inverse relation $p = 1/q$ (note that this is equivalent to the assumption $\lambda = -\gamma$) when $\theta > 0$. This is illustrated in Table 4. Event though depreciation rates are in higher level than in Table 3, the relative increase is small. The effect is not additive with respect to parameters λ and γ. For instance, $-\gamma = \lambda = 0.5$ yields much lower depreciation rate than $\lambda = 0.1$.

Table 4
Results with $\alpha = 1/3$, $\theta = 0.5$, $\lambda = -\gamma$

		R (years)	d (%)	$\frac{d-\delta}{d}$ (%)
$\delta = 0.02 (A > 0)$	$\lambda = 0.03$	14.8	3.77	46.98
	$\lambda = 0.05$	10.1	4.52	55.77
$\delta = 0.04$ $(A > 0)$	$\lambda = 0.03$	12.8	5.93	32.57
	$\lambda = 0.05$	9.2	6.69	40.19

Finally, the last question is addressed. Which one is more substantial: physical depreciation or obsolescence? The proportion of obsolescence to (total) depreciation varies a lot among different scenarios. For instance, the ratio of obsolescence to depreciation varies from 30% to 56% in Tables 3 and 4. The sample mean is 42%, hence physical depreciation seems to be slightly more substantial than obsolescence in contrast to some studies (e.g. Sakellaris & Wilson, 2004, p.3-4, [4]). However, the answer depends heavily on factors of the context such as a type of industry, a type of capital and so on. For example, in ICT sector physical depreciation rates and the growth of I-S technological progress may differ much from more traditional sectors. To study this more closely, parameters are matched to yearly data of the United Kingdom ICT sector. Following Bakhshi and Larsen, δ and α are set to 0.239 and 0.028[24], respectively (Bakhshi & Larsen, 2001,[26]). The steady-state growth rate of investments-specific technological change is set to 18.9%. The results are reported for both cases, either $\gamma = -0.189$ or $\lambda = 0.189$. Since there are no available data for θ, two different scenarios are considered, either $\theta = 0$ or $\theta = 0.3$. Typically, disposed ICT assets have very low market value, thus $\theta = 0$ can be regarded as a reasonable benchmark case. The results are summarized in Table 5. The ratio of obsolescence to depreciation varies from 20.79% to 33.87%, while the sample mean being 27.21%. It can be again concluded that physical depreciation seems to be more substantial than obsolescence. However, the conclusion depends heavily on the estimates of parameter δ. For example, if physical depreciation is set to a lower value $\delta = 0.1$, then obsolescence explains over a half of depreciation given the parameters in the first line of Table 5. So, the question arise whether high physical depreciation estimate $\delta = 0.239$ also includes obsolescence?

[24]The low estimate for capital share α does not affect on results. The results remain almost same when it is chosen $\alpha = 1/3$ (and $r = 0.1$).

Table 5
Results with $\alpha = 0.028$, $\delta = 0.239$

		R (years)	d (%)	$\frac{d-\delta}{d}$ (%)
$\theta = 0$ ($A > 0$)	λ = 0.189	4.5	36.14	33.87
	γ = -0.189	4.5	31.12	23.21
$\theta = 0.3$ ($A > 0$)	λ = 0.189	3.9	34.61	30.95
	γ = -0.189	4.1	30.17	20.79

A few general observations can be made based on the results of the quantitative exercises. First, an increase in the value of scrapped capital stock, that is in θ, accelerates the replacement of capital, that is R. Intuitively this is clear because the absolute cost of replacement is lower (the rebate of scrapped capital is higher) for higher values of θ, thus leading to more frequent capital replacement. Yet, the effect of θ on depreciation d is not so evident. There are two opposing channels: higher θ decreases R, hence increases d, on the other hand, higher θ decreases the loss generated by scrapping (scrapped capital is more valuable) at the moment of replacement, hence decreases d. The second observation is that physical depreciation accelerates capital replacement. This is an expectable outcome since higher δ leads to faster physical depreciation of existing capital stock, hence faster fall in the productivity of existing capital stock, resulting a higher incentive to producer replace earlier. The effect is more prominent what it comes to depreciation d, because in addition to indirect impact through R, δ also increases d directly.

In summary, based on the quantitative study it is likely that all Propositions 1 - 7 except Proposition 5 would hold even if $\theta > 0$. Mukoyama's (2008) model and the generalized model have similar qualitative features, but the levels of variables implied by the models differ. The proportion of obsolescence to depreciation depends heavily on parameter values, especially on the value of physical depreciation. A few illustrative scenarios are studied, and the proportion of physical depreciation is slightly greater than the proportion of obsolescence in most of the cases. However, if physical depreciation is relative low with respect to I-S technological change, then obsolescence will be more substantial than physical depreciation.

6 Implications of the model: the law of motion for capital

In this section it is briefly studied how the results fit into a broader context of macroeconomic modeling. Especially, it is considered how investments-specific technological change can be taken into account in the law of motion for capital. In a world with investment-specific technological change, the following form[25] of the law of motion for capital is commonly found in the macroeconomic literature (e.g. Fisher, 2006,[21]; Deli, 2016, [18]; Greenwood et al., 1988 & 1997, [16] & [5]),

$$\dot{K}(t) = q(t)I(t) - d(t)K(t), \tag{32}$$

[25]The continuous time counterpart is considered here.

where variables have standard meanings. That is, $K(t)$ and $I(t)$ denote aggregate capital stock and aggregate investment, respectively, at time t. Depreciation rate at time t is denoted by $d(t)$ and I-S technological level at time t is denoted by $q(t)$.

There arise two questions concerning the rate of depreciation: "What does the value of d should reflect in this context and how it should be determined?". To answer these, the underlying idea will be to exploit the analogy of PIM method to the law of motion for capital. Then, it will be argued in favor of Mukoyama's interpretation on these questions. Mukoyama applies the depreciation formula (6) in the law of motion for capital (Mukoyama, 2008, p.520, [2]). This implies (in order the application to be valid) that d should reflect both physical depreciation and obsolescence, as well as the formula for depreciation, which is derived from the replacement problem, is appropriate in that context.

The discrete time version of the law of motion for capital closely relates to Perpetual Inventory Method. From the discrete version[26] of Eq. (32) it is easy to show by induction that (given the convention $\prod_{i=t+1}^{t}(1-d_i)=1$),

$$K_{t+1} = \underbrace{\left(\prod_{i=0}^{t}(1-d_i)\right)K_0}_{\text{"depreciation corrected initial capital"}} + \underbrace{\sum_{j=0}^{t}\left(\prod_{i=j+1}^{t}(1-d_i)\right)q_j I_j}_{\text{"depreciation corrected investment flows"}} . \quad (33)$$

This resembles PIM method introduced earlier. In fact, Eq. (33) leads to exactly same equation as PIM method (1), if it is assumed that $q_i = 1$ for all i and $d_i = d_j$ for all i, j. Again, this corresponds to the case in which there is no I-S technological change and depreciation is a constant over time. Nevertheless, there is a profound difference between PIM method (1) and the law of motion for capital. When it is assumed that $q_i = 1$ for all i, it can be seen that depreciation rates $(d_i)_{i=0}^{t}$ relate to investments (of different vintages) $(I_i)_{i=0}$ quite differently in Eq. (1) than in Eq. (33). In PIM method, d_i affects only on investment I_i, hence it reflects "the rate of depreciation of the capital of vintage i". On contrary, investment I_i is evenly affected by all depreciation rates up to period i in the law of motion for capital. Thus, depreciation rates are not vintage specific but rather they reflect an inter-temporal variation of "general" depreciation across all vintages of capital. In Hill's terms (1999), only "*cross section depreciation*" is included in PIM method, but the law of motion for capital also includes the second component of "*time series depreciation*", that is the "*revaluation term*" (Hill, 1999, [10]). Cross section depreciation and the revaluation term correspond to physical depreciation and obsolescence, respectively (Hill, 1999, [10],p.11). From that viewpoint, d in the law of motion for capital, (33), should also include obsolescence.

Why PIM method interpreted as Eq. (1) does not include "the revaluation term", that is obsolescence? The reason is that obsolescence is differently captured in PIM method, namely it is modeled by the retirement distribution. Recall that the retirement distribution is excluded in the formula (1). Based on this, it will be argued that the exclusion of obsolescence in the law of motion for capital is more or less analogous to that the retirement distributions is ignored in Perpetual Inventory Method.

Let us assume that depreciation can be decomposed into physical depreciation and

[26] That is $K_{t+1} = (1-d_t)K_t + q_t I_t$.

obsolescence, and the information about I-S technological change entirely determines[27] obsolescence. In that case the fact $q_i = 1$ for all i (that is, there is no I-S technological change) implies that deprecation is exclusively resulted from physical depreciation, that is $d_i = \delta_i$ for all i. Therefore, instead of assuming $d_i = d_j$ for all i, j, it can be equivalently assumed that $\delta_i = \delta_j$ for all i, j. Now, recall that Eq. (1) and Eq. (33) are equal whenever $d_i = d_j$ and $q_i = 1$. The conclusion is that: if physical depreciation is a constant over time (e.g. due to investments are homogeneous and utilization rates does not vary) and there is no obsolescence (since there is no I-S technological change), then K_t, which is determined by the law of motion for capital, can be equivalently characterized by PIM method in which depreciation function is geometric and retirement distribution is excluded. Therefore, the exclusion of obsolescence in the law of motion for capital is more or less analogous to that the retirement distributions is ignored in PIM method.

Until now, it is argued that obsolescence should be included in the law of motion for capital. The second question concerns the determination of d. There is an interesting implication due to the analogy of obsolescence and the retirement distribution. Namely, the information about the replacement policy is precisely what is needed in order to obsolescence would be appropriately captured, because the replacement policy can be identified with the retirement distribution as discussed in subsection 5.3. From that ground, the replacement problem appears as a natural approach to the determination of depreciation. The replacement problem is addressed in Sections 4 - 5 and some assumptions are imposed along the way. There is the assumption of Cobb-Douglas production technology (see (v)), an assumption of that tthe interest rate is "sufficiently" high relative to I-S technological change (see Proposition 1)[28] and a kind of steady-state assumption (in the sense that a number of plants is a constant in the economy). Given that these assumptions hold, the introduced formula for depreciation, (28), can be used in the law of motion for capital in line with Mukoyama (Mukoyama, 2008, p.520, [2]). For simplicity, let us confine ourselves in the case that scrapped capital stock has no value ($\theta = 0$). Then, by substituting Eq. (27) into Eq. (28) and after some simplifications due to $\theta = 0$, the law of motion for capital can be described as,

$$\dot{K}(t) = q(t)I(t) - d(t)K(t)$$

$$s.t. \quad \begin{cases} d(t) = \dfrac{e^{-\delta R(t)} q(t)^{\frac{\alpha}{1-\alpha}} \left(\dfrac{p(t)}{1 - e^{-(r+\delta\alpha)R(t)}} \right)^{\frac{1}{\alpha-1}}}{\displaystyle\int_0^{R(t)} e^{-\delta s} q(t + R(t) - s)^{\frac{\alpha}{1-\alpha}} \left(\dfrac{p(t + R(t) - s)}{1 - e^{-(r+\delta\alpha)R(t+R(t)-s)}} \right)^{\frac{1}{\alpha-1}} ds} + \delta \\ t \mapsto R(t) \; is \; determined \; by \; the \; family \; of \; difference \; equations: \\ G(T_0 + \sum_{i=0}^{t-1} R_i, R_t, R_{t+1}) = 0, \quad for \; all \; T_0 \in [0, \infty] \end{cases} .$$

Equivalently, if $\theta > 0$, then the depreciation rate can be described by Eqs. (27) and (28)

[27] This assumption in our framework can be formulated as: "$q' = 0$ implies $\theta = 1$". This is a plausible assumption since θ does not reflect physical depreciation and by the depreciation formula (28), it can be seen that depreciation equals to physical depreciation in that case.

[28] For general p and q, the condition $r > (\gamma - \lambda)\frac{\alpha}{\alpha-1}$ may be insufficient. However, the condition is still appropriate, if the following asymptotic results hold:

$$\lim_{t\to\infty} \frac{q(t)}{e^{\lambda t}} = 0 \quad and \quad \lim_{t\to\infty} \frac{e^{\gamma t}}{p(t)} = 0$$

together with the function G.

What if one decides to determine I-S technological change exclusively by q? Then the interpretation of p is not so evident. This closely relates to the issue concerning the relationship of p with q discussed earlier. Due to the endogenity of prices (at least in the general equilibrium context), the price of capital can be thought as a function of I-S technology, that is $p(t) = p(t, q(t))$. Furthermore, the function is typically decreasing with respect to q, that is $\frac{\partial p}{\partial q} < 0$. A potential candidate for the function p is the inverse relation, $p = 1/q$, which is supported by Proposition 4 when $\theta = 0$ and by several studies (e.g. Greenwood et al., 1997, [5]; Hulten, 1992, [24]; Bakhshi & Larsen, 2001, [26]). Therefore, one may assume that $p = 1/q$, which leads to the following description of the law of motion for capital in which there is no explicit mention of the price of capital,

$$\dot{K}(t) = q(t)I(t) - d(t)K(t)$$

$$s.t. \quad \begin{cases} d(t) = \dfrac{e^{-\delta R(t)} q(t)^{\frac{1+\alpha}{1-\alpha}} \left(1 - e^{-(r+\delta\alpha)R(t)}\right)^{\frac{1}{1-\alpha}}}{\displaystyle\int_0^{R(t)} e^{-\delta s} q(t+R(t)-s)^{\frac{1+\alpha}{1-\alpha}} \left(1 - e^{-(r+\delta\alpha)R(t+R(t)-s)}\right)^{\frac{1}{1-\alpha}} ds} + \delta \\ \textit{the family of difference equations } (\tilde{G}_{T_0})_{T_0 \in [0,\infty]} \textit{ determines } t \mapsto R(t) \end{cases}, \qquad (34)$$

where $\tilde{G}_{T_0}$ is derived from the function G in Appendix M,

$$\tilde{G}_{T_0}(R_t, R_{t+1}) \stackrel{\text{def}}{=} \frac{r + \alpha\delta e^{-(r+\delta\alpha)R_{t+1}}}{r + \delta\alpha} - \left(r + \frac{2q'(T_t + R_t)}{q(T_t + R_t)}\right) \frac{(1 - e^{-(r+\delta\alpha)R_{t+1}})\alpha}{r + \alpha\delta}$$
$$- \left(\frac{(1 - e^{-(r+\delta\alpha)R_t}) q(T_t)^2}{(1 - e^{-(r+\delta\alpha)R_{t+1}}) q(T_t + R_t)^2}\right)^{\frac{\alpha}{1-\alpha}} e^{-\delta\alpha R_t} = 0,$$

where T_t is determined by $T_t = T_{t-1} + R_{t-1}$.

The determination of the path of $t \mapsto R(t)$ appears to be a tedious task. However, at the cost of accuracy, we can rest on a linear interpolation of $(R_t)_{t=0}^{\infty}$ as proposed earlier. In that case, the difference equation $\tilde{G}_0(R_t, R_{t+1}) = 0$ must be solved, and then Eq. (29) can be used to obtain $t \mapsto R(t)$.

For illustrative purpose, let us consider the following economic environment. There is a steady growth in I-S technology, say it grows about at rate λ per year, but there is no neutral technological change. The price of capital is also reflected on the progress of I-S technology, say the price of capital falls about at rate λ per year. Scrapped capital has no value, the rate of physical depreciation δ is known and the interest rate[29] r satisfies $r > \frac{2\lambda\alpha}{1-\alpha}$, where α is the share of capital in production. Then, by Lemma 1 and Formula (30), the law of motion for capital can be described as,

$$\dot{K}(t) = e^{\lambda t} I(t) - dK(t)$$

$$s.t. \quad \begin{cases} d = \frac{\delta + \frac{1+\alpha}{1-\alpha}\lambda}{e^{(\delta + \frac{1+\alpha}{1-\alpha}\lambda)R} - 1} + \delta \\ e^{-(\alpha\delta - \frac{2\alpha\lambda}{\alpha-1})R} - \frac{\alpha(r+\delta+2\lambda)}{r+\alpha\delta} e^{-(r+\alpha\delta)R} - \frac{r - \alpha(r+2\lambda)}{r+\alpha\delta} = 0 \end{cases}.$$

Here, deprecation d is a constant over time, because I-S technology has a simple evolution (a steady exponential growth). In that case the value of d can be just calculated and then substituted into the law of motion for capital. However, if the evolution of I-S technology was more complex or otherwise different, then d wouldn't be necessary a constant, consequently d must be determined by Eq. (34).

[29]The risk premium must be also included.

7 Conclusion

Capital stock can be endogenized in Mukoyama's (2008) model such that the stationarity of the optimal replacement policy is retained. When the evolution of I-S technological change deviates from a steady constant rate growth, then the optimal replacement policy may not be stationary, which in turn implies non-stationary depreciation rate. It is proposed a method for the determination of depreciation in that case. The determination of non-stationary depreciation is computationally inconvenient. In stationary case, the replacement policy can be easily determined. Even though there does not exist a closed-from solution for the optimal replacement interval, it can be solved as a root of a relative simple transcendental function. The corresponding (constant) depreciation rate can be calculated by the formula that is equivalent to the formula proposed by Mukoyama (2008). The only difference is that the change in the growth rate of the productivity of a new capital is also taken into account.

I-S technological progress can described either as a fall in the price of capital or as a growth in the relative productivity of new capital. From the viewpoint of the optimal replacement policy, these approaches are equivalent as long as scrapped capital stock has no value and I-S technological change grows at constant rate. An increase in the growth rate of I-S technology, interpreted as an increase either in the growth of productivity or in the fall of prices, leads to more frequent capital replacement. This in turn implies higher depreciation due replacement (that is, higher obsolescence).

The adoption of the capital replacement problem for describing depreciation is a promising approach. The major advantage is the consistency with the producers optimization, hence the applicability of the results is easier to justify in a broader context. In particular, it is argued that the depreciation rate can be applied in the law of motion for capital that is almost ubiquitous in macroeconomics.

The model does not include labor and neutral technological change is not taken into account. However, the results are robust to a number of parameters, in particular to the level of neural technology (the parameter A). Physical depreciation seems to be a substantial part of depreciation, hence the rate of depreciation depends heavily on the estimate of physical depreciation. The existence of the replacement policy requires that the interest rate must be "sufficiently" high relative to I-S technological change. This is because the budget constraint is not included. An important task that remains to be done, is to include the budget constraint in order to get rid of the interest rate condition. Further, an interesting future research topic is to study what is the impact of neutral technological progress on the replacement decision, or does the inclusion of labor has an effect on the optimal replacement policy.

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A Appendix: The derivation of Mukoyama's model

Let assume that the producer maximizes the present value of its profits. The production technology is described by the production function $O(K(t,s))$, which depends only on one input, the amount of capital at time t whose age is s. The evolution of the capital stock differs quite lot from standard characterization encountered in macroeconomics. Firstly, there is no capital accumulation. When producer decides to replace its capital stock, it is assumed that producer always invest in the frontier quality capital and the fraction θ of existing capital stocks is retrieved back to the producer. Until next replacement, the plant production is based on capital replaced in that period. Secondly, the producer has only one control, to decide the frequency of capital replacement.

At first, a bit more general model is derived and then it is reduced back to the Mukoyama's (2008) model. The producer has to choose a sequence of times, when the capital is replaced, say $(T_i)_{i=1}^{\infty} = (T_1, T_2, T_3, ...)$. Then the discounted output from the period $]T_t, T_{t+1}]$, whose capital has been just installed, is defined by the formula,

$$\int_0^{T_{t+1}-T_t} e^{-rs} O(T_t, s) ds = \int_0^{\Delta T_{t+1}} e^{-rs} O(T_t, s) ds, \tag{35}$$

where $r > 0$ denotes a discount rate. At the end of the period, an investment in new capital is exogenously given. Its cost to the producer is,

$$p(T_{t+1}) k(T_{t+1}, 0). \tag{36}$$

On the other hand, the fraction θ of existing capital is retrieved back to the producer,

$$\theta p(T_{t+1}) k(T_t, \Delta T_{t+1}). \tag{37}$$

Now, the terms (35)- (37) are exploited in presenting the replacement problem. The value of plant at time t is denoted by a function $V(t)$. The value of plant is determined by the discounted flow of profits. Let us consider the value of plant at the beginning of the period $]T_t, T_{t+1}]$, that is, at time T_t,

$$V(T_t) = \max_{\{T_i\}_{i=t+1}^{\infty}} \sum_{i=t}^{\infty} \Bigg(e^{-r(T_i - T_t)} \int_0^{\Delta T_{i+1}} e^{-rs} O(T_i, s) ds \tag{38}$$

$$+ e^{-r(T_{i+1}-T_t)} \Big(-p(T_{i+1}) k(T_{i+1}, 0) + \theta p(T_{i+1}) k(T_i, \Delta T_{i+1}) \Big) \Bigg). \tag{39}$$

Next, we modify the equation in such that the value of plant is represented in terms of current profits and the future value of plant. Extract the first term from the series,

$$V(T_t) = \max_{\{T_i\}_{i=t+1}^{\infty}} e^{-r(T_t - T_t)} \int_0^{\Delta T_{t+1}} e^{-rs} O(T_t, s) ds$$
$$+ e^{-r(T_{t+1}-T_t)} \Big(-p(T_{t+1}) k(T_{t+1}, 0) + \theta p(T_{t+1}) k(T_t, \Delta T_{t+1}) \Big)$$
$$+ \sum_{i=t+1}^{\infty} \Bigg(e^{-r(T_i - T_t)} \int_0^{\Delta T_{i+1}} e^{-rs} O(T_i, s) ds$$
$$+ e^{-r(T_{i+1}-T_t)} \Big(-p(T_{i+1}) k(T_{i+1}, 0) + \theta p(T_{i+1}) k(T_i, \Delta T_{i+1}) \Big) \Bigg).$$

Observe that $e^{-r(T_t-T_t)}=1$ and $e^{-r(T_{t+1}-T_t)}=e^{-r\Delta T_{t+1}}$,

$$V(T_t)=\max_{\{T_i\}_{i=t+1}^{\infty}}\int_0^{\Delta T_{t+1}}e^{-rs}O(T_t,s)ds$$
$$+e^{-r\Delta T_{t+1}}\Big(-p(T_{t+1})k(T_{t+1},0)+\theta p(T_{t+1})k(T_t,\Delta T_{t+1})\Big)$$
$$+\sum_{i=t+1}^{\infty}\Big(e^{-r(T_i-T_t)}\int_0^{\Delta T_{i+1}}e^{-rs}O(T_i,s)ds$$
$$+e^{-r(T_{i+1}-T_t)}\Big(-p(T_{i+1})k(T_{i+1},0)+\theta p(T_{i+1})k(T_i,\Delta T_{i+1})\Big)\Big).$$

Note that $e^{T_{t+2}-T_t}=e^{T_{t+2}-T_{t+1}+T_{t+1}-T_t}=e^{T_{t+2}-T_{t+1}+\Delta T_{t+1}}$, etc...

$$V(T_t)=\max_{\{T_i\}_{i=t+1}^{\infty}}\int_0^{\Delta T_{t+1}}e^{-rs}O(T_t,s)ds$$
$$+e^{-r\Delta T_{t+1}}\Big(-p(T_{t+1})k(T_{t+1},0)+\theta p(T_{t+1})k(T_t,\Delta T_{t+1})\Big)$$
$$+e^{-r\Delta T_{t+1}}\sum_{i=t+1}^{\infty}\Big(e^{-r(T_i-T_{t+1})}\int_0^{\Delta T_{i+1}}e^{-rs}O(T_i,s)ds$$
$$+e^{-r(T_{i+1}-T_{t+1})}\Big(-p(T_{i+1})k(T_{i+1},0)+\theta p(T_{i+1})k(T_i,\Delta T_{i+1})\Big)\Big).$$

Then, notice that the last two terms of the right-hand side are equal to the value of plant at time T_{t+1}[30],

$$V(T_t)=\max_{\Delta T_{t+1}}\int_0^{\Delta T_{t+1}}e^{-rs}O(T_t,s)ds$$
$$+e^{-r\Delta T_{t+1}}\Big(-p(T_{t+1})k(T_{t+1},0)+\theta p(T_{t+1})k(T_t,\Delta T_{t+1})\Big)+e^{-r\Delta T_{t+1}}V(T_{t+1}).$$

In the Mukoyama's model it is assumed that "*the economy is in the steady-state, in the sense that... the replacement of capital occurs at same rate every period*" (Mukoyama, 2008, p.518, [2]). In our framework that means there exists a constant $T>0$ such that $\Delta T_t=T\ for\ all\ t$. To see the identicality with Mukoyama's (2008) model, denote $t=T_t$ and note that $T_{t+1}=T_{t+1}-T_t+T_t=\Delta T_{t+1}+T_t=T+t$. In that case the replacement problem reads as,

$$V(t)=\max_T\int_0^T e^{-rs}O(t,s)ds+e^{-rT}\Big(-p(t+T)k(t+T,0)+\theta p(t+T)k(t,T)+V(t+T)\Big).$$

[30] In the period T_t the producer has to choose T_{t+1}, that is same as to choose ΔT_{t+1} for given T_t.

B Appendix: An alternative solution of Mukoyama's model

The following question is studied: Would the Mukoyama's model (2008) be still solvable if some of assumptions (i)-(iv) were relaxed? The answer will be tentative yes. The result foreshadows even greater relaxation of the assumptions that is taken in Section 4.

At this moment, it is only imposed some regularity conditions on the functional forms of output, capital and capital price. More precisely, let's relax assumptions (ii) - (iv) in the following sense[31]

(viii). $k(t,0) \in C^1(\mathbb{R}_+)$

(ix). $O(t,s) \in C(\mathbb{R}_+^2)$

(x). $p(t) \in C^1(\mathbb{R}_+)$

Our strategy is to convert the functional equation (3) into an ordinary differential equation. It turns out that obtained ordinary differential equation is a first order linear ODE, which is easy to solve. To convert the equation, we first notice that the necessary condition for the existence of a local maximum in the right-hand side of the equation (3) is that the derivative with respect to T should vanish,

$$0 = \frac{\partial}{\partial T}\Big(\int_0^T e^{-rs} O(t,s) ds + e^{-rT}\Big(V(t+T) - p(t+T)k(t+T,0) + \theta p(t+T)k(t,T)\Big)\Big). \tag{40}$$

By the assumption (ix), a function $e^{-rs}O(t,s)$ *is continuous on* $[0,\infty[$ *in variable* s. Therefore, the Fundamental Theorem of Calculus guarantees,

$$\frac{\partial}{\partial T}\int_0^T e^{-rs}O(t,s)ds = e^{-rT}O(t,T). \tag{41}$$

Besides the second term takes a form,

$$\begin{aligned} &\frac{\partial}{\partial T}\Big(e^{-rT}\Big(V(t+T) - p(t+T)k(t+T,0) + \theta p(t+T)k(t,T)\Big)\Big) \\ &= -re^{-rT}\Big(V(t+T) - p(t+T)k(t+T,0) + \theta p(t+T)k(t,T)\Big) \\ &+ e^{-rT}\Big(\frac{\partial V(t+T)}{\partial T} - \frac{\partial p(t+T)}{\partial T}k(t+T,0) - p(t+T)\frac{\partial k(t+T,0)}{\partial T} \\ &+ \theta\frac{\partial p(t+T)}{\partial T}k(t,T) + \theta p(t+T)\frac{\partial k(t,T)}{\partial T}\Big). \end{aligned} \tag{42}$$

Substitute Eqs. (41) and (42) into Eq. (40) and multiply by the factor e^{rT} to get,

$$rV(t+T) - \frac{\partial V(t+T)}{\partial T} = h(t,T), \tag{43}$$

[31]The class $C^1(X)$ consists of all continuously differentiable functions on a domain X

where we denote,

$$
\begin{aligned}
&h(t,T) \stackrel{\text{def}}{=} O(t,T) + rp(t+T)k(t+T,0) \\
&-\frac{\partial p(t+T)}{\partial T}k(t+T,0) - p(t+T)\frac{\partial k(t+T,0)}{\partial T} \\
&-\theta\Big(rp(t+T)k(t,T) - \frac{\partial p(t+T)}{\partial T}k(t,T) - p(t+T)\frac{\partial k(t,T)}{\partial T}\Big).
\end{aligned}
$$

Let us clean a bit Eq. (43). First, notice that by the chain rule, the partial derivatives can be represented as,[32]

$$
\begin{aligned}
&\frac{\partial V(t+T)}{\partial T} = \frac{\partial(t+T)}{\partial T}\frac{\partial V}{\partial t}(t+T) = V'(t+T) \\
&\frac{\partial k(t+T,0)}{\partial T} = \frac{\partial(t+T)}{\partial T}\frac{\partial k}{\partial t}(t+T,0) = \frac{\partial k}{\partial t}(t+T,0) \\
&\frac{\partial k(t,T)}{\partial T} = \frac{\partial k}{\partial s}(t,T) \\
&\frac{\partial p(t+T)}{\partial T} = \frac{\partial(t+T)}{\partial T}\frac{\partial p}{\partial t}(t+T) = p'(t+T).
\end{aligned}
$$

Now, by the change of variables, $x = t+T$, Eq. (42) can be represented as,

$$
V'(x) - rV(x) = -h(x-T,T). \tag{44}
$$

This is just a linear first order ODE without an initial value condition $V(x_0)$[33]. There exists a solution if the function $-h(x-T,T)$ is continuous (in fact only integrability is required). The assumptions (i) and (viii) - (x) guarantee the continuity of the function $-h(x-T,T)$. Solving the ODE (44) is straightforward and we solve it by using standard techniques.

The integrating factor is,

$$
\mu(x) = e^{\int_{x_0}^{x}(-r)ds} = e^{r(x_0-x)}, \quad i.e. \quad \mu'(x) = -r\mu(x).
$$

Multiply both sides of Eq. (44) by the integrating factor and rearrange terms to get,

$$
\begin{aligned}
&\mu(x)V'(x) - r\mu(x)V(x) = -\mu(x)h(x-T,T) \\
&\Leftrightarrow \quad \mu(x)V'(x) + \mu'(x)V(x) = -\mu(x)h(x-T,T) \\
&\Rightarrow \quad \frac{\partial}{\partial x}\Big(\mu(x)V(x)\Big) = -\mu(x)h(x-T,T).
\end{aligned}
$$

[32] The following notations are adopted:
$\frac{\partial k}{\partial t}(t+T,0) = \frac{\partial k(t,s)}{\partial t}\Big|_{(t,s)=(t+T,0)}$
$\frac{\partial k}{\partial s}(t,T) = \frac{\partial k(t,s)}{\partial s}\Big|_{(t,s)=(t,T)}$
etc...

[33] $x_0 = t + T_0 = t + 0 = t$ *and the value* $V(t)$ *is unknown*

Integrate both sides of the equation over the interval $[x_0, x]$,

$$\int_{x_0}^{x} \frac{\partial}{\partial z}\Big(\mu(z)V(z)\Big)dz = -\int_{x_0}^{x} \mu(z)h(z-T,T)dz$$
$$\mu(x)V(x) - \mu(x_0)V(x_0) = -\int_{x_0}^{x} \mu(z)h(z-T,T)dz.$$

to obtain,

$$V(x) = \frac{\mu(x_0)V(x_0)}{\mu(x)} - \frac{1}{\mu(x)}\int_{x_0}^{x} \mu(z)h(z-T,T)dz. \tag{45}$$

Notice that $\mu(x_0) = e^{r(x_0-x_0)} = 1$. By substituting $x = t+T$ back to the solution of $V(x)$ we get,

$$V(t+T) = V(t)e^{rT} - e^{rT}\int_{t}^{t+T} e^{r(t-z)}h(z-T,T)dz.$$

For optimal T, this equation can be substituted back to the original problem without maximum operator, that is to equation,

$$V(t) = \int_0^T e^{-rs}O(t,s)ds$$
$$+ e^{-rT}\Big(V(t+T) - p(t+T)k(t+T,0) + \theta p(t+T)k(t,T)\Big).$$

And we obtain,

$$G(t,T) \overset{\text{def}}{=} \int_0^T e^{-rs}O(t,s)ds - \int_t^{t+T} e^{r(t-z)}h(z-T,T)dz \tag{46}$$
$$- e^{-rT}p(t+T)k(t+T,0) + e^{-rT}\theta p(t+T)k(t,T) = 0.$$

This equation, $G(t,T) = 0$, characterizes the optimal replacement interval T. The result is analogous to Mukoyama's result (5) (Mukoyama, 2008, [2]). The main difference is that in general case the solution is not necessarily stationary, that is, the solution depends on time t. But now, it is not so clear whether the implicit equation (46) has a solution for all reasonable parameter values and for all times t or whether the solutions is unique.

Since Eq. (46) is the solution to the model, the result can be verified easily. Mukoyama calibrates his model and computes magnitude of effect on T when changing different parameter values. By assuming same functional forms (i.e. assumptions (i) - (iv)) and setting same parameter values, one should get same results. Matlab exercise verifies that results are same for parameter values described in tables 1-3 in Mukoyama's paper.

C Appendix

Proposition.

$$\frac{(1-\theta)\frac{k(t-T,T)}{T}}{\int_0^T \frac{k(t-s,s)}{T}ds} + \delta = \frac{(1-\theta)\lambda}{e^{\lambda T}-1} + \delta, \quad where\ \lambda \overset{\text{def}}{=} \delta + \frac{\gamma}{1-\alpha}$$

Proof. By the assumption (i) we have the following,

$$\frac{(1-\theta)\frac{k(t-T,T)}{T}}{\int_0^T \frac{k(t-s,s)}{T}ds} = \frac{(1-\theta)k(t-T,T)}{\int_0^T k(t-s,s)ds} = \frac{(1-\theta)e^{-\delta T}k(t-T,0)}{\int_0^T e^{-\delta s}k(t-s,0)ds}.$$

By the assumption (ii) this becomes,

$$\frac{(1-\theta)e^{-\delta T}k(t-T,0)}{\int_0^T e^{-\delta s}k(t-s,0)ds} = \frac{(1-\theta)e^{-\delta T}e^{\frac{1}{1-\alpha}\gamma(t-T)}}{\int_0^T e^{-\delta s}e^{\frac{1}{1-\alpha}\gamma(t-s)}ds}$$
$$= \frac{(1-\theta)e^{-\delta T}e^{\frac{1}{1-\alpha}\gamma(t-T)}}{\int_0^T e^{-s(\delta+\frac{1}{1-\alpha}\gamma)}e^{\frac{1}{1-\alpha}\gamma t}ds} = \frac{(1-\theta)e^{-\delta T}e^{\frac{1}{1-\alpha}\gamma(t-T)}}{\left(-\frac{1}{\delta+\frac{1}{1-\alpha}\gamma}e^{-s(\delta+\frac{1}{1-\alpha}\gamma)}\right)\Big|_0^T e^{\frac{1}{1-\alpha}\gamma t}}$$
$$= \frac{(1-\theta)e^{-\delta T}e^{\frac{1}{1-\alpha}\gamma(t-T)}}{\left(\frac{1}{\delta+\frac{1}{1-\alpha}\gamma} - \frac{1}{\delta+\frac{1}{1-\alpha}\gamma}e^{-T(\delta+\frac{1}{1-\alpha}\gamma)}\right)e^{\frac{1}{1-\alpha}\gamma t}}.$$

And by denoting $\lambda \stackrel{\text{def}}{=} \delta + \frac{\gamma}{1-\alpha}$ this simplifies to,

$$\frac{(1-\theta)e^{-\delta T}e^{\frac{1}{1-\alpha}\gamma(t-T)}}{\left(\frac{1}{\delta+\frac{1}{1-\alpha}\gamma} - \frac{1}{\delta+\frac{1}{1-\alpha}\gamma}e^{-T(\delta+\frac{1}{1-\alpha}\gamma)}\right)e^{\frac{1}{1-\alpha}\gamma t}} = \frac{(1-\theta)e^{-\delta T}e^{(\lambda-\delta)(t-T)}}{\left(\frac{1}{\lambda}-\frac{1}{\lambda}e^{-T\lambda}\right)e^{(\lambda-\delta)t}}$$
$$= \frac{(1-\theta)e^{-\delta T}e^{(\lambda-\delta)t}e^{-T\lambda}e^{T\delta}}{\left(\frac{1}{\lambda}-\frac{1}{\lambda}e^{-T\lambda}\right)e^{(\lambda-\delta)t}} = \frac{(1-\theta)e^{-T\lambda}}{\left(\frac{1}{\lambda}-\frac{1}{\lambda}e^{-T\lambda}\right)}$$
$$= \frac{(1-\theta)e^{-T\lambda}}{\frac{1}{\lambda}e^{-T\lambda}\left(e^{T\lambda}-1\right)} = \frac{(1-\theta)}{\frac{1}{\lambda}\left(e^{T\lambda}-1\right)} = \frac{(1-\theta)\lambda}{e^{T\lambda}-1}.$$

□

D Appendix

Let us deduce the value of capital stock, which is installed at time T_t, but evaluated at time T_{t+1}. The value of capital stock equals to the value of investment $p(T_t)I_t$ at time T_t, further just calculate,

$$p(T_t)I_t = p(T_t)I_t\frac{q(T_t)}{q(T_t)} = p(T_t)\frac{k(T_t,0)}{q(T_t)}.$$

At time T_{t+1}, the age of capital stock is $T_{t+1} - T_t = \Delta T_{t+1}$. Moreover, the price of capital has changed from $p(T_t)$ to $p(T_{t+1})$. Thus, the value of capital stock, which is installed at time T_t, but evaluated at time T_{t+1} is,

$$p(T_{t+1})\frac{k(T_t,\Delta T_{t+1})}{q(T_t)}.$$

Furthermore, it is important to notice that besides the value of capital stock declines due to physical depreciation (and possibly due to the decline of capital prices), the value of capital stock also declines in the opportunity cost sense. At the time T_{t+1} there is available more productive capital in the capital markets due to the investment-specific technological progress.

E Appendix

$$\begin{aligned}
&\frac{\partial}{\partial s}k(t,s) = -\delta k(t,s) \\
\Leftrightarrow \quad & \frac{\partial}{\partial s}k(t,s) + \delta k(t,s) = 0 \\
\Leftrightarrow \quad & e^{\delta s}\frac{\partial}{\partial s}k(t,s) + e^{\delta s}\delta k(t,s) = 0 \\
\Leftrightarrow \quad & \frac{\partial}{\partial s}\Big(e^{\delta s}k(t,s)\Big) = 0 \\
\Leftrightarrow^{34} & \int_0^s \frac{\partial}{\partial u}\Big(e^{\delta u}k(t,u)\Big)du = 0 \\
\Leftrightarrow \quad & e^{\delta s}k(t,s) - k(t,0) = 0 \\
\Leftrightarrow \quad & k(t,s) = e^{-\delta s}k(t,0)
\end{aligned}$$

Using the initial value condition $k(T_t,0) = q(T_t)I_t$ in (7), we have:

$$k(T_t,s) = e^{-\delta s}q(T_t)I_t$$

F Appendix

The first term of the series can be extracted, so the series can be written in the form,

$$\begin{aligned}
&V\big(T_t,K_t\big) = \max_{(R_i)_{i=t}^{\infty},(I_i)_{i=t+1}^{\infty}} \Bigg\{ e^{-r(T_t-T_t)}\int_0^{R_t} e^{-rs}O(T_t,s)ds \\
&+ e^{-r(T_{t+1}-T_t)}\Big(-p(T_{t+1})I_{t+1} + \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)}\Big) \\
&+ \sum_{i=t+1}^{\infty}\Bigg(e^{-r(T_i-T_t)}\int_0^{R_i} e^{-rs}O(T_i,s)ds \\
&+ e^{-r(T_{i+1}-T_t)}\Big(-p(T_{i+1})I_{i+1} + \theta p(T_{i+1})\frac{e^{-\delta R_i}K_i}{q(T_i)}\Big)\Bigg)\Bigg\}.
\end{aligned}$$

[34] The fact, $k(t,s) \geq 0$ *for all* (t,s), is exploited.

Observe that $e^{-r(T_t-T_t)}=1$ and $e^{-r(T_{t+1}-T_t)}=e^{-rR_t}$, which leads to,

$$V\big(T_t,K_t\big) = \max_{(R_i)_{i=t}^{\infty},(I_i)_{i=t+1}^{\infty}} \Bigg\{ \int_0^{R_t} e^{-rs}O(T_t,s)ds$$
$$+ e^{-rR_t}\Big(-p(T_{t+1})I_{t+1} + \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)}\Big)$$
$$+ \sum_{i=t+1}^{\infty}\Bigg(e^{-r(T_i-T_t)}\int_0^{R_i} e^{-rs}O(T_i,s)ds$$
$$+ e^{-r(T_{i+1}-T_t)}\Big(-p(T_{i+1})I_{i+1} + \theta p(T_{i+1})\frac{e^{-\delta R_i}K_i}{q(T_i)}\Big)\Bigg)\Bigg\}.$$

Note that $e^{T_{t+i}-T_t}=e^{T_{t+i}-T_{t+1}+T_{t+1}-T_t}=e^{T_{t+i}-T_{t+1}+R_t}$, hence the series becomes,

$$V\big(T_t,K_t\big) = \max_{(R_i)_{i=t}^{\infty},(I_i)_{i=t+1}^{\infty}} \Bigg\{ \int_0^{R_t} e^{-rs}O(T_t,s)ds$$
$$+ e^{-rR_t}\Big(-p(T_{t+1})I_{t+1} + \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)}\Big)$$
$$+ e^{-rR_t}\sum_{i=t+1}^{\infty}\Bigg(e^{-r(T_i-T_{t+1})}\int_0^{R_i} e^{-rs}O(T_i,s)ds$$
$$+ e^{-r(T_{i+1}-T_{t+1})}\Big(-p(T_{i+1})I_{i+1} + \theta p(T_{i+1})\frac{e^{-\delta R_i}K_i}{q(T_i)}\Big)\Bigg)\Bigg\}.$$

Represent the maximum operator such that it is split into two parts,

$$V\big(T_t,K_t\big) = \max_{R_t,I_{t+1}} \Bigg\{ \int_0^{R_t} e^{-rs}O(T_t,s)ds$$
$$+ e^{-rR_t}\Big(-p(T_{t+1})I_{t+1} + \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)}\Big)$$
$$+ e^{-rR_t}\max_{(R_i)_{i=t+1}^{\infty},(I_i)_{i=t+2}^{\infty}} \Bigg\{ \sum_{i=t+1}^{\infty}\Bigg(e^{-r(T_i-T_{t+1})}\int_0^{R_i} e^{-rs}O(T_i,s)ds$$
$$+ e^{-r(T_{i+1}-T_{t+1})}\Big(-p(T_{i+1})I_{i+1} + \theta p(T_{i+1})\frac{e^{-\delta R_i}K_i}{q(T_i)}\Big)\Bigg)\Bigg| R_t, I_{t+1} \textit{ are given}\Bigg\}\Bigg\}.$$

Then, notice that the term enclosed by the second maximum operator is same as the value of plant at time T_{t+1} with initial capital K_{t+1}. We obtain,

$$V\big(T_t,K_t\big) = \max_{R_t,I_{t+1}} \int_0^{R_t} e^{-rs}O(T_t,s)ds$$
$$+ e^{-rR_t}\Big(-p(T_{t+1})I_{t+1} + \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)}\Big)$$
$$+ e^{-rR_t}V(T_{t+1},K_{t+1}).$$

Rearrange last terms. The replacement problem can be represented as Bellman equation,

$$V\big(T_t, K_t\big) = \max_{R_t, I_{t+1}} \int_0^{R_t} e^{-rs} O(T_t, s) ds$$
$$+ e^{-rR_t}\Big(- p(T_{t+1})I_{t+1} + \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)} + V(T_{t+1}, K_{t+1})\Big).$$

G Appendix

The first-order condition for the control variable R_t reads as,

$$AK_t^{\alpha}e^{-(r+\delta\alpha)R_t} - re^{-rR_t}\Big(- p(T_{t+1})I_{t+1} + \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)} + V(T_{t+1}, K_{t+1})\Big)$$
$$+ e^{-rR_t}\Big(- p'(T_{t+1})I_{t+1} + \theta p'(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)} - \delta\theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)}$$
$$+ \frac{\partial}{\partial T_{t+1}}V(T_{t+1}, K_{t+1}) + \frac{\partial}{\partial K_{t+1}}V(T_{t+1}, K_{t+1})q'(T_{t+1})I_{t+1}\Big) = 0.$$

From the first-order condition, the partial derivative $\frac{\partial}{\partial T_{t+1}}V(T_{t+1}, K_{t+1})$ can be solved. After lagging the result by one period,

$$\frac{\partial}{\partial T_t}V(T_t, K_t) = -AK_{t-1}^{\alpha}e^{-\delta\alpha R_{t-1}} + r\Big(- p(T_t)I_t + \theta p(T_t)\frac{e^{-\delta R_{t-1}}K_{t-1}}{q(T_{t-1})} + V(T_t, K_t)\Big)$$
$$+ p'(T_t)I_t - \theta p'(T_t)\frac{e^{-\delta R_{t-1}}K_{t-1}}{q(T_{t-1})} + \delta\theta p(T_t)\frac{e^{-\delta R_{t-1}}K_{t-1}}{q(T_{t-1})} - \frac{p(T_t)}{q(T_t)}q'(T_t)I_t.$$

Inserting these obtained forms of $\frac{\partial}{\partial T_t}V(T_t, K_t)$ and $\frac{\partial}{\partial T_{t+1}}V(T_{t+1}, K_{t+1})$ and also $\frac{\partial}{\partial K_{t+1}}V(T_{t+1}, K_{t+1})$ (Eq. (19)) into the costate equation of variable T_t, one gets a quite complex formula. But it turns out that terms $\pm\frac{p(T_{t+1})}{q(T_{t+1})}q'(T_{t+1})I_{t+1}$, $\pm p'(T_{t+1})I_{t+1}$ and $\pm\theta p'(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)}$ cancels out, which results a simpler formula,

$$- AK_{t-1}^{\alpha}e^{-\delta\alpha R_{t-1}} + r\Big(- p(T_t)I_t + \theta p(T_t)\frac{e^{-\delta R_{t-1}}K_{t-1}}{q(T_{t-1})} + V(T_t, K_t)\Big)$$
$$+ p'(T_t)I_t - \theta p'(T_t)\frac{e^{-\delta R_{t-1}}K_{t-1}}{q(T_{t-1})} + \delta\theta p(T_t)\frac{e^{-\delta R_{t-1}}K_{t-1}}{q(T_{t-1})} - \frac{p(T_t)}{q(T_t)}q'(T_t)I_t$$
$$= e^{-rR_t}\Bigg(- \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)^2}q'(T_t) - AK_t^{\alpha}e^{-\delta\alpha R_t} + \delta\theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)}$$
$$+ r\Big(- p(T_{t+1})I_{t+1} + \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)} + V(T_{t+1}, K_{t+1})\Big)\Bigg).$$

To get rid of the value functions appearing in the equation, the maximized Bellman equation should be exploited. The maximized Bellman equation is the original Bellman equation (16) given that controls are chosen optimally(in this case maximal operator disappears). From this, the difference of the discounted value of plant in two consequent periods, $V(T_t, K_t) - e^{-rR_t}V(T_{t+1}, K_{t+1})$, is easy to compute[35] and using this one obtains,

$$
\begin{aligned}
&- AK_{t-1}^{\alpha}e^{-\delta\alpha R_{t-1}} + rAK_t^{\alpha}\frac{1-e^{-(r+\delta\alpha)R_t}}{r+\delta\alpha} + re^{-rR_t}\Big(- p(T_{t+1})I_{t+1} + \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)}\Big) \\
&+ r\Big(- p(T_t)I_t + \theta p(T_t)\frac{e^{-\delta R_{t-1}}K_{t-1}}{q(T_{t-1})}\Big) \\
&+ p'(T_t)I_t - \theta p'(T_t)\frac{e^{-\delta R_{t-1}}K_{t-1}}{q(T_{t-1})} + \delta\theta p(T_t)\frac{e^{-\delta R_{t-1}}K_{t-1}}{q(T_{t-1})} - \frac{p(T_t)}{q(T_t)}q'(T_t)I_t \\
&= e^{-rR_t}\Bigg(- \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)^2}q'(T_t) - AK_t^{\alpha}e^{-\delta\alpha R_t} + \delta\theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)})\Bigg) \\
&+ re^{-rR_t}\Big(- p(T_{t+1})I_{t+1} + \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)}\Big).
\end{aligned}
$$

The terms $re^{-rR_t}\Big(- p(T_{t+1})I_{t+1} + \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)}\Big)$ cancels out. After forwarding the result by one period, we end up with the function G,

$$
\begin{aligned}
&r\Bigg(AK_{t+1}^{\alpha}\frac{1-e^{-(r+\delta\alpha)R_{t+1}}}{r+\delta\alpha} - p(T_{t+1})I_{t+1} + \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)}\Bigg) - AK_t^{\alpha}e^{-\delta\alpha R_t} \\
&+ p'(T_{t+1})I_{t+1} - \theta p'(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)} + \delta\theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)} - \frac{p(T_{t+1})}{q(T_{t+1})}q'(T_{t+1})I_{t+1} \\
&= e^{-rR_{t+1}}\Bigg(- \theta p(T_{t+2})\frac{e^{-\delta R_{t+1}}K_{t+1}}{q(T_{t+1})^2}q'(T_{t+1}) - AK_{t+1}^{\alpha}e^{-\delta\alpha R_{t+1}} + \delta\theta p(T_{t+2})\frac{e^{-\delta R_{t+1}}K_{t+1}}{q(T_{t+1})}\Bigg).
\end{aligned}
$$

H Appendix

Assume that $\theta = 0$. Then the function G read as,

$$
\begin{aligned}
G(T_t, R_t, R_{t+1}) &= rAK_{t+1}^{\alpha}\frac{1-e^{-(r+\delta\alpha)R_{t+1}}}{r+\delta\alpha} - rp(T_{t+1})I_{t+1} - AK_t^{\alpha}e^{-\delta\alpha R_t} \\
&+ p'(T_{t+1})I_{t+1} - \frac{p(T_{t+1})}{q(T_{t+1})}q'(T_{t+1})I_{t+1} + e^{-(r+\alpha\delta)R_{t+1}}AK_{t+1}^{\alpha}.
\end{aligned}
$$

[35]

$$
V(T_t, K_t) - e^{-rR_t}V(T_{t+1}, K_{t+1}) = AK_t^{\alpha}\frac{1-e^{-(r+\delta\alpha)R_t}}{r+\delta\alpha} + e^{-rR_t}\Big(- p(T_{t+1})I_{t+1} + \theta p(T_{t+1})\frac{e^{-\delta R_t}K_t}{q(T_t)}\Big)
$$

After substituting the expressions I_t, K_t, q and p into the previous equation and rearranging the terms, the function G can be written as,

$$
\begin{aligned}
&-Ae^{-\alpha\delta R_t}\left(\frac{(\alpha\delta+r)e^{T_t(\gamma-\lambda)}}{\alpha A\left(1-e^{R_t(-(\alpha\delta+r))}\right)}\right)^{\frac{\alpha}{\alpha-1}}+\frac{Ar\left(\frac{(\alpha\delta+r)e^{(\gamma-\lambda)(R_t+T_t)}}{\alpha A\left(1-e^{R_{t+1}(-(\alpha\delta+r))}\right)}\right)^{\frac{\alpha}{\alpha-1}}}{\alpha\delta+r}\\
&+\frac{\alpha A\delta e^{R_{t+1}(-(\alpha\delta+r))}\left(\frac{(\alpha\delta+r)e^{(\gamma-\lambda)(R_t+T_t)}}{\alpha A\left(1-e^{R_{t+1}(-(\alpha\delta+r))}\right)}\right)^{\frac{\alpha}{\alpha-1}}}{\alpha\delta+r}\\
&+(\gamma-r-\lambda)e^{(\gamma-\lambda)(R_t+T_t)}\left(\frac{(\alpha\delta+r)e^{(\gamma-\lambda)(R_t+T_t)}}{\alpha A\left(1-e^{R_{t+1}(-(\alpha\delta+r))}\right)}\right)^{\frac{1}{\alpha-1}}.
\end{aligned}
$$

Differentiating with respect to T_t, yields for all terms a coefficient $(\gamma-\lambda)\frac{\alpha}{\alpha-1}$. Thus $\frac{\partial}{\partial T_t}G$ equals to,

$$
\begin{aligned}
&-\frac{\alpha(\gamma-\lambda)}{(\alpha-1)(\alpha\delta+r)}\left(A(\alpha\delta+r)e^{-\alpha\delta R_t}\left(\frac{(\alpha\delta+r)e^{T_t(\gamma-\lambda)}}{\alpha A\left(1-e^{R_t(-(\alpha\delta+r))}\right)}\right)^{\frac{\alpha}{\alpha-1}}\right.\\
&-Ar\left(\frac{(\alpha\delta+r)e^{(\gamma-\lambda)(R_t+T_t)}}{\alpha A\left(1-e^{R_{t+1}(-(\alpha\delta+r))}\right)}\right)^{\frac{\alpha}{\alpha-1}}\\
&-\alpha A\delta e^{R_{t+1}(-(\alpha\delta+r))}\left(\frac{(\alpha\delta+r)e^{(\gamma-\lambda)(R_t+T_t)}}{\alpha A\left(1-e^{R_{t+1}(-(\alpha\delta+r))}\right)}\right)^{\frac{\alpha}{\alpha-1}}\\
&\left.-(\alpha\delta+r)(\gamma-r-\lambda)e^{(\gamma-\lambda)(R_t+T_t)}\left(\frac{(\alpha\delta+r)e^{(\gamma-\lambda)(R_t+T_t)}}{\alpha A\left(1-e^{R_{t+1}(-(\alpha\delta+r))}\right)}\right)^{\frac{1}{\alpha-1}}\right).
\end{aligned}
$$

It can be seen that the following equation holds,

$$\frac{\partial}{\partial T_t}G=(\gamma-\lambda)\frac{\alpha}{\alpha-1}G.$$

I Appendix

Suppose $\theta=0$. By Proposition 2 we can assume that $T_t=0$. Further, by exploiting the equation: $T_{t+1}=T_t+R_t$, the function G reads as,

$$
\begin{aligned}
G(R_t,R_{t+1})={}&rAK_{t+1}^{\alpha}\frac{1-e^{-(r+\delta\alpha)R_{t+1}}}{r+\delta\alpha}-rp(R_t)I_{t+1}\\
&-AK_t^{\alpha}e^{-\delta\alpha R_t}+p'(R_t)I_{t+1}-\frac{p(R_t)}{q(R_t)}q'(R_t)I_{t+1}+e^{-rR_{t+1}}AK_{t+1}^{\alpha}e^{-\delta\alpha R_{t+1}}
\end{aligned}
$$

Substituting the expressions of investment, capital stock, capital price and investment-specific technological change and assuming that there exists $R > 0$ such that $R_t = R = R_{t+1}$,

$$G(0,R,R) = \frac{Ar\left(-\frac{(\alpha\delta+r)e^{R(\gamma-\lambda)}}{\alpha A\left(e^{-R(\alpha\delta+r)}-1\right)}\right)^{\frac{\alpha}{\alpha-1}}}{\alpha\delta+r} + \frac{\alpha A\delta e^{-R(\alpha\delta+r)}\left(-\frac{(\alpha\delta+r)e^{R(\gamma-\lambda)}}{\alpha A\left(e^{-R(\alpha\delta+r)}-1\right)}\right)^{\frac{\alpha}{\alpha-1}}}{\alpha\delta+r}$$
$$+ (\gamma-r-\lambda)e^{R(\gamma-\lambda)}\left(-\frac{(\alpha\delta+r)e^{R(\gamma-\lambda)}}{\alpha A\left(e^{-R(\alpha\delta+r)}-1\right)}\right)^{\frac{1}{\alpha-1}} - Ae^{-\alpha\delta R}\left(-\frac{\alpha\delta+r}{\alpha A\left(e^{-R(\alpha\delta+r)}-1\right)}\right)^{\frac{\alpha}{\alpha-1}}.$$

Divide the equation $G(0,R,R)=0$ by $\left(\frac{-1}{A\alpha(e^{-(r+\alpha\delta)R}-1)}\right)^{\frac{\alpha}{\alpha-1}}$,

$$\frac{Ar\left((\alpha\delta+r)e^{R(\gamma-\lambda)}\right)^{\frac{\alpha}{\alpha-1}}}{\alpha\delta+r} + \frac{\alpha A\delta e^{-R(\alpha\delta+r)}\left((\alpha\delta+r)e^{R(\gamma-\lambda)}\right)^{\frac{\alpha}{\alpha-1}}}{\alpha\delta+r}$$
$$+ (\gamma-r-\lambda)(r+\alpha\delta)^{\frac{1}{\alpha-1}}e^{(\gamma-\lambda+\frac{\gamma-\lambda}{\alpha-1})R}\left(\frac{-1}{A\alpha(e^{-(r+\alpha\delta)R}-1)}\right)^{-1} - Ae^{-\alpha\delta R}(\alpha\delta+r)^{\frac{\alpha}{\alpha-1}} = 0.$$

Note that $e^{(\gamma-\lambda+\frac{\gamma-\lambda}{\alpha-1})R}\left(\frac{-1}{A\alpha(e^{-(r+\alpha\delta)R}-1)}\right)^{-1}$ equals to $A\alpha e^{((\gamma-\lambda)\frac{\alpha}{\alpha-1})R} - A\alpha e^{((\gamma-\lambda)\frac{\alpha}{\alpha-1}-(r+\alpha\delta))R}$. After dividing the equation by $-(r+\alpha\delta)^{\frac{\alpha}{\alpha-1}}$ we conclude that,

$$Ae^{((\gamma-\lambda)\frac{\alpha}{\alpha-1})R}\left(e^{-(\alpha\delta+(\gamma-\lambda)\frac{\alpha}{\alpha-1})R} - \frac{\alpha(r+\delta+\lambda-\gamma)}{r+\alpha\delta}e^{-(r+\alpha\delta)R} - \frac{r+\alpha(\gamma-\lambda-r)}{r+\alpha\delta}\right) = 0.$$

J Appendix

We know that the optimal investments at times $(T_i)_{i=1}^{\infty}$ are given by Eq. (21),

$$I_t = \frac{1}{q(T_t)}\left(\frac{\frac{p(T_t)}{q(T_t)} - \theta e^{-rR_t}p(T_{t+1})\frac{e^{-\delta R_t}}{q(T_t)}}{\alpha A\frac{1-e^{-(r+\delta\alpha)R_t}}{r+\delta\alpha}}\right)^{\frac{1}{\alpha-1}},$$

and the capital stock installed at time T_t whose age is s, is given by $k(T_t,s) = e^{-\delta s}q(T_t)I_t$. A quantity $T_{t+1} = T_t + R_t$ refers to coming capital replacement time, and hence for arbitrary time t the counterpart of T_{t+1} is $t + R(t)$. The extension is based on the assumption that investments $t \mapsto I(t)$ can determined by Eq. 21 outside of the set $R(T_i)_{i=0}^{\infty}$

Then, given the path of optimal replacement times $\left(R(t)\right)_{t\geq 0}$, the investments can be extended onto positive real line,

$$I(t) = \frac{1}{q(t)}\left(\frac{\frac{p(t)}{q(t)} - \theta e^{-rR(t)}p(t+R(t))\frac{e^{-\delta R(t)}}{q(t)}}{\alpha A\frac{1-e^{-(r+\delta\alpha)R(t)}}{r+\delta\alpha}}\right)^{\frac{1}{\alpha-1}}.$$

Correspondingly, the capital stock can be continuously determined as,

$$k(t,s) = e^{-\delta s} q(t) I(t) = e^{-\delta s} \left(\frac{\frac{p(t)}{q(t)} - \theta e^{-rR(t)} p(t+R(t)) \frac{e^{-\delta R(t)}}{q(t)}}{\alpha A \frac{1-e^{-(r+\delta\alpha)R(t)}}{r+\delta\alpha}} \right)^{\frac{1}{\alpha-1}}.$$

K Appendix

Mukoyama's formula reads as,

$$d = \frac{(1-\theta)\frac{k(t-T,T)}{T}}{\int_0^T \frac{k(t-s,s)}{T} ds} + \delta.$$

The variable T corresponds to the function $R(t)$ in our context,

$$d = \frac{(1-\theta)\frac{k(t-R(t),R(t))}{R(t)}}{\int_0^{R(t)} \frac{k(t-s,s)}{R(t)} ds} + \delta.$$

The terms of the form $k(t-s,s)$ must be replaced by $\frac{k(t,s)}{q(t)}$ due to the same reason as explained in the Appendix D,

$$d = \frac{(1-\theta)\frac{k(t-R(t),R(t))}{q(t-R(t))}\frac{1}{R(t)}}{\int_0^{R(t)} \frac{k(t-s,s)}{q(t-s)}\frac{1}{R(t)} ds} + \delta.$$

The timing in the formula is a bit inconvenient. To see this note that at time $t=0$ it holds $t-T=-T<0$, hence leading to undefined or meaningless quantity $k(t-T,T)$, at least in our context. Therefore, consider depreciation evaluated at the end of the period ($[t, t+R(t)]$), that is at $t+R(t)$, rather than at the beginning of the period, that is at t,

$$d = \frac{(1-\theta)\frac{k(t+R(t)-R(t),R(t))}{q(t+R(t)-R(t))}\frac{1}{R(t)}}{\int_0^{R(t)} \frac{k(t+R(t)-s,s)}{q(t+R(t)-s)}\frac{1}{R(t)} ds} + \delta = \frac{(1-\theta)\frac{k(t,R(t))}{q(t)}\frac{1}{R(t)}}{\int_0^{R(t)} \frac{k(t+R(t)-s,s)}{q(t+R(t)-s)}\frac{1}{R(t)} ds} + \delta.$$

Thus, a convenient way to define depreciation rate is,

$$d(t) \stackrel{\text{def}}{=} \frac{(1-\theta)\frac{k(t,R(t))}{q(t)}\frac{1}{R(t)}}{\int_0^{R(t)} \frac{k(t+R(t)-s,s)}{q(t+R(t)-s)}\frac{1}{R(t)} ds} + \delta.$$

L Appendix

Assume that $T_0 = 0$ and $R_t = R$. Then, $T_t = tR$ and $T_{t+s} = (s+t)R$. Hence,

$$I_t = \frac{1}{q(T_t)} \left(\frac{\frac{p(T_t)}{q(T_t)} - \theta e^{-rR_t} p(T_{t+1}) \frac{e^{-\delta R_t}}{q(T_t)}}{\alpha A \frac{1-e^{-(r+\delta\alpha)R_t}}{r+\delta\alpha}} \right)^{\frac{1}{\alpha-1}}$$

takes a form,

$$I_t = \frac{1}{q(tR)} \left(\frac{\frac{p(tR)}{q(tR)} - \theta e^{-rR} p((t+1)R) \frac{e^{-\delta R}}{q(tR)}}{\alpha A \frac{1-e^{-(r+\delta\alpha)R}}{r+\delta\alpha}} \right)^{\frac{1}{\alpha-1}}$$

By using the assumptions (vii) and (vi), we obtain,

$$I_t = e^{-\lambda tR} \left(\frac{(r+\alpha\delta) e^{-(\delta+r+t\lambda)R} \left(e^{(\delta+r+\gamma t)R} - \theta e^{\gamma(t+1)R} \right)}{\alpha A \left(1 - e^{-(r+\alpha\delta)R}\right)} \right)^{\frac{1}{\alpha-1}}.$$

The trick is to define $I(t) \stackrel{\text{def}}{=} I_{\frac{t}{R}}$ in order to get a function of continuous variable t (measured in years). Thus,

$$I(t) = e^{-\lambda t} \left(\frac{(r+\alpha\delta) e^{-(\delta+r)R+\lambda t} \left(e^{(\delta+r)R+\gamma t} - \theta e^{\gamma(t+R)} \right)}{\alpha A \left(1 - e^{-(r+\alpha\delta)R}\right)} \right)^{\frac{1}{\alpha-1}}.$$

Recall that,

$$d(t) = \frac{(1-\theta)e^{-\delta R} I(t)}{\int_0^R e^{-\delta s} I(t+R-s) ds} + \delta.$$

The integral in the denominator is surprisingly complex to calculate. Even for a computer algebra system[36], the integration may take several minutes. Nevertheless, the formula simplifies to a simple expression, which is independent of time t,

$$d = \frac{(1-\theta)((1-\alpha)\delta + \alpha\lambda - \gamma)}{(1-\alpha)\left(e^{(\delta + \frac{\alpha\lambda-\gamma}{1-\alpha})R} - 1\right)} + \delta,$$

which can be also written in the form,

$$d = \frac{(1-\theta)(\delta + \frac{\alpha\lambda-\gamma}{1-\alpha})}{e^{(\delta+\frac{\alpha\lambda-\gamma}{1-\alpha})R} - 1} + \delta.$$

[36] Wolfram Mathematica 11.2 was used in symbolic integration.

M Appendix

When $\theta = 0$, then the function G read as,

$$G(T_t, R_t, R_{t+1}) = rAK_{t+1}^{\alpha}\frac{1-e^{-(r+\delta\alpha)R_{t+1}}}{r+\delta\alpha} - rp(T_{t+1})I_{t+1} - AK_t^{\alpha}e^{-\delta\alpha R_t}$$
$$+ p'(T_{t+1})I_{t+1} - \frac{p(T_{t+1})}{q(T_{t+1})}q'(T_{t+1})I_{t+1} + e^{-(r+\alpha\delta)R_{t+1}}AK_{t+1}^{\alpha}.$$

By exploiting the assumption $p = 1/q$ yields,

$$G(T_t, R_t, R_{t+1}) = rAK_{t+1}^{\alpha}\frac{1-e^{-(r+\delta\alpha)R_{t+1}}}{r+\delta\alpha} - \frac{rI_{t+1}}{q(T_{t+1})} - AK_t^{\alpha}e^{-\delta\alpha R_t}$$
$$- \frac{q'(T_{t+1})I_{t+1}}{q(T_{t+1})^2} - \frac{q'(T_{t+1})I_{t+1}}{q(T_{t+1})^2} + e^{-(r+\alpha\delta)R_{t+1}}AK_{t+1}^{\alpha}.$$

Using the fact $I_{t+1} = \frac{K_{t+1}}{q(T_{t+1})}$, this becomes,

$$G(T_t, R_t, R_{t+1}) = rAK_{t+1}^{\alpha}\frac{1-e^{-(r+\delta\alpha)R_{t+1}}}{r+\delta\alpha} - \frac{r\frac{K_{t+1}}{q(T_{t+1})}}{q(T_{t+1})} - AK_t^{\alpha}e^{-\delta\alpha R_t}$$
$$- \frac{q'(T_{t+1})\frac{K_{t+1}}{q(T_{t+1})}}{q(T_{t+1})^2} - \frac{q'(T_{t+1})\frac{K_{t+1}}{q(T_{t+1})}}{q(T_{t+1})^2} + e^{-(r+\alpha\delta)R_{t+1}}AK_{t+1}^{\alpha}$$
$$= rAK_{t+1}^{\alpha}\frac{1-e^{-(r+\delta\alpha)R_{t+1}}}{r+\delta\alpha} - \frac{rK_{t+1}}{q(T_{t+1})^2} - AK_t^{\alpha}e^{-\delta\alpha R_t}$$
$$- \frac{2q'(T_{t+1})K_{t+1}}{q(T_{t+1})^3} + e^{-(r+\alpha\delta)R_{t+1}}AK_{t+1}^{\alpha}.$$

Since $(R_t)_{t=0}^{\infty}$ is characterized by the roots of the function G, it can be equivalently confined to solve the roots of the function $\frac{K_{t+1}^{-\alpha}}{A}G(T_t, R_t, R_{t+1})$,

$$\frac{K_{t+1}^{-\alpha}}{A}G(T_t, R_t, R_{t+1}) = \frac{r - re^{-(r+\delta\alpha)R_{t+1}}}{r+\delta\alpha} - \Big(r + \frac{2q'(T_{t+1})}{q(T_{t+1})}\Big)\frac{K_{t+1}^{1-\alpha}}{Aq(T_{t+1})^2}$$
$$- \Big(\frac{K_t}{K_{t+1}}\Big)^{\alpha}e^{-\delta\alpha R_t} + e^{-(r+\alpha\delta)R_{t+1}}.$$

Thus, to find the optimal $(R_t)_{t=0}^{\infty}$ is reduced to solving the roots of the following equation,

$$\frac{r + \alpha\delta e^{-(r+\delta\alpha)R_{t+1}}}{r+\delta\alpha} - \Big(r + \frac{2q'(T_{t+1})}{q(T_{t+1})}\Big)\frac{K_{t+1}^{1-\alpha}}{Aq(T_{t+1})^2} - \Big(\frac{K_t}{K_{t+1}}\Big)^{\alpha}e^{-\delta\alpha R_t} = 0.$$

When $\theta = 0$ and $p = 1/q$, then the optimal K_t takes a form,

$$K_t = \left(\frac{\alpha A(1-e^{-(r+\delta\alpha)R_t})q(T_t)^2}{r+\alpha\delta}\right)^{\frac{1}{1-\alpha}}.$$

By substituting this into the equation and using the fact $T_{t+1} = T_t + R_t$, we obtain the final result. The optimal $(R_t)_{t=0}^{\infty}$ is characterized by the roots of the following difference equation,

$$\tilde{G}_{T_0}(R_t, R_{t+1}) \stackrel{\text{def}}{=} \frac{r + \alpha\delta e^{-(r+\delta\alpha)R_{t+1}}}{r + \delta\alpha} - \left(r + \frac{2q'(T_t + R_t)}{q(T_t + R_t)}\right)\frac{(1 - e^{-(r+\delta\alpha)R_{t+1}})\alpha}{r + \alpha\delta}$$
$$- \left(\frac{(1 - e^{-(r+\delta\alpha)R_t})q(T_t)^2}{(1 - e^{-(r+\delta\alpha)R_{t+1}})q(T_t + R_t)^2}\right)^{\frac{\alpha}{1-\alpha}} e^{-\delta\alpha R_t} = 0,$$

where initial value T_0 varies over interval $[0, \infty[$.